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2006 J. Phys. A: Math. Gen. 39 13921

(http://iopscience.iop.org/0305-4470/39/45/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 13921–13950 doi:10.1088/0305-4470/39/45/005

Generating functional analysis of batch minority games with arbitrary strategy numbers

N Shayeghi and A C C Coolen

Department of Mathematics, King's College London, The Strand, London WC2R 2LS, UK

E-mail: nima@mth.kcl.ac.uk and ton.coolen@kcl.ac.uk

Received 12 June 2006, in final form 19 September 2006
Published 24 October 2006
Online at stacks.iop.org/JPhysA/39/13921

Abstract

Both the phenomenology and the theory of minority games (MG) with more than two strategies per agent are different from those of the conventional and extensively studied case $S = 2$. MGs with $S > 2$ exhibit nontrivial statistics of the frequencies with which the agents select from their available decision making strategies, with far-reaching implications. In the few theoretical MG studies with $S > 2$ published so far, these statistics could not be calculated analytically. This prevented solution even in ergodic stationary states; equations for order parameters could only be closed approximately, using simulation data. Here we carry out a generating functional analysis of fake history batch MGs with arbitrary values of S, and give an analytical solution of the strategy frequency problem. This leads to closed equations for order parameters in the ergodic regime, exact expressions for strategy selection statistics, and phase diagrams. Our results find perfect confirmation in numerical simulations.

PACS numbers: 02.50.Le, 87.23.Ge, 05.70.Ln, 64.60.Ht

1. Introduction

Minority games [1, 2] were proposed as simple models with which to increase our understanding of the origin of the observed nontrivial dynamics of markets; these dynamics are believed to result from an interplay of cooperation, competition and adaptation of interacting agents. Minority games were found to exhibit intriguing nontrivial relations between observables such as the market volatility, overall bid correlations and sensitivity to market perturbations, and the information available for agents to act upon. Still, it was found that they can to a large extent be solved analytically using equilibrium and non-equilibrium statistical mechanical techniques (see e.g., the recent textbooks [3, 4] and references therein). This combination of complexity and solvability is their beauty and appeal. There have been many advances in the mathematical study of MGs, but nearly all calculations have been carried out for $S = 2$, where S is the number of trading strategies available to each agent.

One typically finds phase transitions separating an ergodic from a non-ergodic regime, which for $S = 2$ can be located exactly. For $S > 2$, in contrast, no such exact results are available. Although mathematical theory has been developed along the lines of the $S = 2$ case [5–8], the authors of the latter studies ultimately ran into the nontrivial problem of calculating the statistics of agents' strategy selections, a problem which resisted analytical solution. In this paper we generalize the theorists' toolbox: we develop a generating functional analysis [9] for MGs with arbitrary finite values of S, and show how to derive exact closed equations for observables and phase transition lines. We do so for the simplest so-called batch version of the MG [10, 11], with fake (random) external information [12].

Our paper is organized as follows. We start with definitions and review the phenomenology of MGs with $S > 2$ as observed in simulations, focusing on the differences with the familiar $S = 2$ case and on the core problem of determining the statistics of strategy selection frequencies. The next step is a formal generating functional analysis (of which most details are relegated to an appendix), leading as usual to an effective single agent process, here describing the evolution of S strategy valuations. We then state the strategy selection frequency problem in the language of the effective agent process, and solve it. The result is an exact but nontrivial set of closed equations for the relevant observables in MGs with arbitrary S in the ergodic phase, and predictions for the location of the phase transition line. $S = 3$ is the simplest nontrivial situation to which our theory applies, so we present extensive applications to $S = 3$ and test each prediction against numerical simulations. This is followed by further (more limited) applications to $S = 4$ and $S = 5$ MGs, again verified via simulation experiments, and predictions regarding market volatility and predictability. Although the theory is initially set up for arbitrary types and levels of decision noise, we concentrate in this paper mostly on MGs with either additive or absent decision noise. We end with a discussion of our results and their implications.

2. Definitions

The MG describes N agents in a market, labelled by $i = 1, \ldots, N$. Each agent i is required to submit a bid $b_i(t) \in \{-1, 1\}$ (e.g., 'sell' or 'buy') to this market, at each round $t \in \{0, 1, 2, \ldots\}$ of the game, in response to public information which is distributed to all agents. Those who subsequently find themselves in the minority group, i.e. those i for which $b_i(t)\big[\sum_j b_j(t)\big] < 0$ (who sell when most wish to buy, or buy when most wish to sell), make profit. In order to be successful in the game, agents must therefore anticipate how their competitors are likely to respond to the public information.

The mathematical implementation of the game is as follows. The public information at time t (e.g., the state of the market) is represented by an integer number $\mu(t) \in \{1, \ldots, p\}$. Each agent i has S private strategies $\mathbf{R}^{ia} \in \{-1, 1\}^p$ at his disposal, labelled by $a = 1, \ldots, S$, with which to convert the observed information into a binary bid. A strategy is a look-up table giving prescribed bids for each of the p possible states of the market: upon observing $\mu(t) = \mu$ at time t, strategy a of agent i would prescribe submitting the bid $b_i(t) = R^{ia}_{\mu}$. All pNS entries $R^{ia}_{\mu} \in \{-1, 1\}$ are drawn randomly and independently at the start of the game, with equal probabilities. The strategy used by agent i at time t is called his 'active strategy', and denoted by $a_i(t) \in \{1, \ldots, S\}$. Given the active strategies of all agents, their collective responses are fully deterministic; the dynamics of the MG evolves around the evolution of the N active strategies $a_i(t)$.

Agents in the MG select their active strategies on the basis of strategy valuations $v_{ia}(t)$, which indicate how often each strategy would have been profitable if it had been played from the start of the game. In the so-called batch version of the MG these valuations are continually updated following

$$v_{ia}(t+1) = v_{ia}(t) + \theta_{ia}(t) - \frac{\tilde{\eta}}{\sqrt{N}} \sum_{\mu=1}^{p} A_\mu(t) R_\mu^{ia}. \tag{1}$$

Here $A_\mu(t) = N^{-1/2} \sum_j R_\mu^{j a_j(t)}$ is the re-scaled overall market bid at time t that would be observed upon presentation of external information μ, $\tilde{\eta}$ (the 'learning rate') is a parameter that controls the characteristic time scales of the process, and $\theta_{ia}(t)$ denotes a (small) perturbation that enables us to define response functions. We abbreviate $\mathbf{v}^i(t) = (v_{i1}(t), \ldots, v_{iS}(t)) \in \mathbb{R}^S$. The active strategies are now determined at each time t and for each agent i by a function $m : \mathbb{R}^S \to \{1, \ldots, S\}$ that promotes strategies with large valuations, but may allow for a degree of randomness. Typical choices are

$$\text{deterministic:} \qquad m(\mathbf{v}) = \mathrm{argmax}_a[v_a] \tag{2}$$

$$\text{additive noise:} \qquad m(\mathbf{v}) = \mathrm{argmax}_a[v_a + T z_a(t)] \tag{3}$$

$$\text{multiplicative noise:} \qquad m(\mathbf{v}) = r(t)\, \mathrm{argmax}_a[v_a] + [1 - r(t)]a(t). \tag{4}$$

Here $z_a(t)$ is a zero-average and unit-variance random variable, drawn independently for each (a, t), $T \geqslant 0$ is a parameter to control the randomness in (3), $r(t) \in \{0, 1\}$ is drawn randomly and independently for each t from some distribution $P(r)$, and $a(t)$ is drawn randomly and independently for each t from $\{1, \ldots, S\}$ (with equal probabilities). We recover the deterministic case (2) by taking $T \to 0$ in (3), or $P(r) \to \delta(r - 1)$ in (4). To compactify subsequent equations we write all random variables at time t in (3), (4), drawn separately and independently for each of the N agents, as $\mathbf{z}(t)$. This allows us to write generally $a_i(t) = m(\mathbf{v}^i(t), \mathbf{z}^i(t))$. The (stochastic) equations describing the MG with arbitrary S can now be written as

$$v_{ia}(t+1) = v_{ia}(t) + \theta_{ia}(t) - \frac{\tilde{\eta}}{N} \sum_{\mu=1}^{p} \sum_{j=1}^{N} R_\mu^{ia} R_\mu^{j m(\mathbf{v}^j(t), \mathbf{z}^j(t))}. \tag{5}$$

Alternatively we can write the process (5) in probabilistic form, i.e. as an evolution equation for the probability density $P(\mathbf{v}^1, \ldots, \mathbf{v}^N)$ of the N strategy valuation vectors. Upon abbreviating $\{\mathbf{z}\} = (\mathbf{z}^1, \ldots, \mathbf{z}^N)$ and $\{\mathbf{v}\} = (\mathbf{v}^1, \ldots, \mathbf{v}^N)$ this can be written as

$$P_{t+1}(\{\mathbf{v}\}) = \int \mathrm{d}\{\mathbf{v}'\} W_t(\{\mathbf{v}\}; \{\mathbf{v}'\}) P_t(\{\mathbf{v}'\}) \tag{6}$$

$$W_t(\{\mathbf{v}\}; \{\mathbf{v}'\}) = \left\langle \prod_{ia} \delta\left[v_{ia} - v'_{ia} - \theta_{ia}(t) + \frac{\tilde{\eta}}{N} \sum_{\mu j} R_\mu^{ia} R_\mu^{j m(\mathbf{v}^{j\prime}, \mathbf{z}^j)} \right] \right\rangle_{\{\mathbf{z}\}}. \tag{7}$$

Averages over the process (6), (7) are written as $\langle \cdots \rangle$. The market fluctuations in the MG are characterized by the average $\langle A(t) \rangle = p^{-1} \sum_\mu \langle A_\mu(t) \rangle$, which one expects to be zero, and the covariance kernel $\Xi_{tt'} = p^{-1} \sum_\mu \langle [A_\mu(t) - \langle A(t) \rangle][A_\mu(t') - \langle A(t') \rangle] \rangle$. In particular, the volatility σ is given by $\sigma^2 = \lim_{\tau\to\infty} \tau^{-1} \sum_{t=1}^{\tau} \Xi_{tt}$, and the predictability is measured by $H = \lim_{\tau\to\infty} \tau^{-2} \sum_{tt'=1}^{\tau} \Xi_{tt'}$. For detailed discussions of the relations between the above and various alternative MG versions we refer to [3, 4].

3. Phenomenology of MGs with arbitrary values of S

Before diving into theory it is helpful to describe first the phenomenology of MGs with $S > 2$, as observed in numerical simulations, with emphasis on those aspects in which they differ from $S = 2$ ones. All simulations in this paper involved MGs with $N = 4097$ agents and no decision noise; stationary state measurements were taken either over the time interval $500 \leqslant t \leqslant 3000$ (for $\alpha < 32$) or over $100 \leqslant t \leqslant 1100$ (for $\alpha \geqslant 32$).

In early numerical and theoretical studies on MGs with $S > 2$ the emphasis was often on the behaviour of the volatility. Its dependence on α was found to be very similar for different values of S, but with the phase transition point $\alpha_c(S)$ (separating a nonergodic regime at small α from an ergodic regime for large α) increasing with S. The differences between $S = 2$ MGs and $S > 2$ MGs become clear as soon as one goes beyond measuring observables derived from the overall market bid (like the volatility), but turns to quantities such as the fraction of 'frozen' agents or the long-time correlations. For $S > 2$ it is no longer obvious how such objects must be defined. Some agents are found to play just one strategy, some play two, some play three, etc; for $S > 2$ agents can apparently be 'frozen' to various extents, which cannot be captured by a single number ϕ (the fraction of frozen agents for $S = 2$, see e.g. [3, 4]). Similarly, it is not *a priori* clear which variables should be measured to define correlation functions.

The dynamics of MGs is about the selection of active strategies $a_i(t) \in \{1, \ldots, S\}$ by the agents, so let us observe in simulations how agents select strategies for $S > 2$ in the stationary state. We define the frequencies $f_a^i = \lim_{\tau\to\infty} \tau^{-1} \sum_{t\leqslant\tau} \delta_{a,a_i(t)}$, where f_a^i measures the fraction of time during which agent i played strategy a. Each vector $\mathbf{f}_i = \left(f_1^i, \ldots, f_S^i\right)$ is a point in the $(S-1)$-dimensional plane $\sum_{a=1}^S f_a^i = 1$ in $[0, 1]^S$. The collection of these N points gives a view on the stationary state of the system. For $S = 3$ the vectors $\mathbf{f}_i$ can be plotted directly as points in $[0, 1]^3$, giving rise to figures such as figure 1. We can extract relevant information from these graphs. For large α the agents tend to involve all three strategies, but not with identical frequencies (otherwise one would have seen $\mathbf{f}_i = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ for all i). As α is reduced, the vectors $\mathbf{f}_i$ tend to concentrate on the borders of the plane $\sum_{a=1}^S f_a^i = 1$ in $[0, 1]^S$, which is where one of the f_a^i equals zero; here agents play only two of their strategies, but not with identical frequencies. Upon reducing α further we enter the nonergodic regime (which explains the difference between the two graphs for $\alpha = 1/64$), with points either concentrating in the corners $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ (for biased initialization) or on more constrained subsets where two strategies are played in very specific combinations (for unbiased initialization). If we measure the distribution $\varrho(f_a) = N^{-1} \sum_i \delta\left(f_a - f_a^i\right)$ we also obtain quantitative information on the *density* of points in various regions of $[0, 1]^3$, complementing figure 1. Typical examples are shown in figure 2, for different values of α; the symmetry of the problem guarantees that for $N \to \infty$ all S distributions $\varrho(f_a)$ must be identical (this is confirmed in simulations). Temporal information on how such macroscopic states are realized can be obtained by showing the variables $a_i(t)$ as functions of time. For $S = 3$ and unbiased initial conditions this has been done using grey-scale coding in figure 3, showing the difference between the small α regime (left), where agents tend to alternate two of their strategies equally, and the large α regime, where agents tend to involve all three strategies, at non-uniform rates.

For $S = 4$ one can no longer plot the frequency vectors $\mathbf{f}_i = \left(f_1^i, f_2^i, f_3^i, f_4^i\right)$ as points in $[0, 1]^3$, but one has to resort to projection: we plot in figure 4 the first three components $\left(f_1^i, f_2^i, f_3^i\right)$. The corresponding strategy frequency distributions are shown in figure 5. Again we observe the ergodicity/nonergodicity phase transition (here occurring for a larger value of α than was the case at $S = 3$), and the tendency at small α for agents to play only a specific subset of their four strategies. For unbiased initial conditions the agents are seen to play in the nonergodic regime always either one or two of their strategies; this curious tendency, for which there is no immediate explanation, is found also for larger values of S (we confirmed

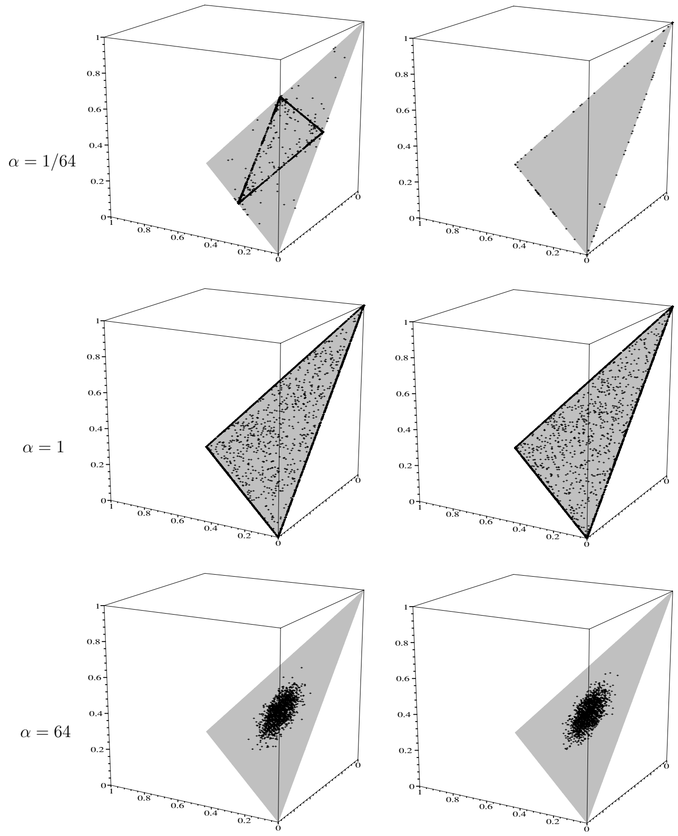

Figure 1. Each plot shows the frequency vectors $\mathbf{f}_i = (f_1^i, f_2^i, f_3^i)$ with which the agents use their three strategies, drawn for each agent i as a point in $[0, 1]^3$ (giving N points per plot), as obtained from numerical simulations of an MG without decision noise with $N = 4097$ and $S = 3$. The constraint $\sum_a f_a^i = 1$ for all i implies that all points are in the plane that goes through the three corners $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ (each corner represents agents 'frozen' into using a single strategy). Left plots: unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$). Right plots: biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$).

this for values up to $S = 6$). As with $S = 3$ we see the agents playing all S strategies for large α, but selecting specific subsets for smaller α, generally with nontrivial frequencies. Here there are more options than at $S = 3$ for doing so: agents can and will go for either one, two, three or four strategies.

Strategy selections are made on the basis of strategy valuations, which in MGs are known to grow potentially linearly with time. Only those strategies with the largest growth rates will be played. We measured the valuation growth rates (or 'strategy velocities') $\overline{v}_a^i = \tau^{-1}\big(v_a^i(t+\tau) - v_a^i(t)\big)$, in the stationary state (i.e. for large τ and t). For small α all velocities are concentrated very close to the origin; as α is increased they become consistently more negative, with concentration of points with two identical components at intermediate α (consistent with agents playing two strategies only) and concentration of points along the diagonal where all three components are identical for large α (consistent with agents playing all three strategies). See e.g., figure 6. If we plot the distribution $\varrho(\overline{v}_a) = N^{-1}\sum_i \delta\big(\overline{v}_a - \overline{v}_a^i\big)$ of the strategy velocities for strategy a, we see the tendency of the strategy velocities to shift

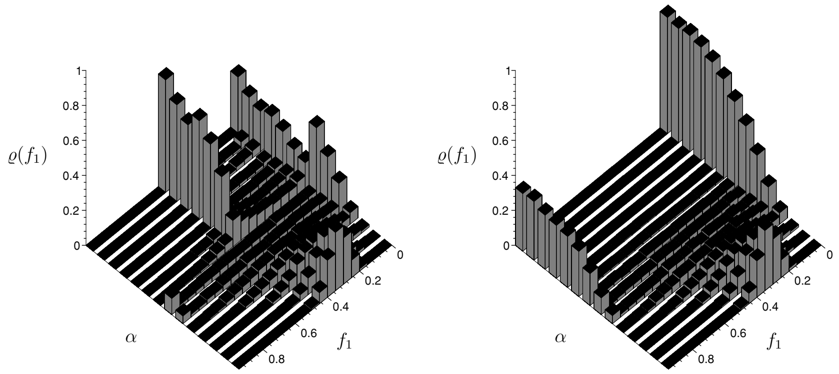

Figure 2. Histograms of the fraction $\varrho(f_1)$ of agents that play strategy 1 with frequency f_1 in the stationary state (observed in simulations with $S = 3$), for $\alpha \in \{1/128, 1/64, \ldots, 32, 64\}$ (increasing by a factor of 2 at each step). Left: unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$). Right: biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$).

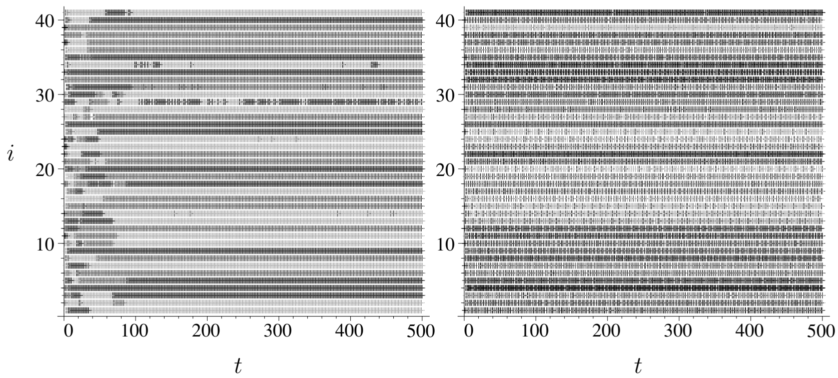

Figure 3. Evolution in time of active strategy selections for the first 41 out of the total number $N = 4097$ of agents in an MG with $S = 3$, following unbiased initial conditions. The chosen strategies are indicated by grey levels: white means $a_i(t) = 1$, grey means $a_i(t) = 2$, and black means $a_i(t) = 3$. Left graph: $\alpha = 1/128$ (in the nonergodic regime). Right graph: $\alpha = 8$ (in the ergodic regime).

towards negative values as α increases (indicating an increasing inability of the agents to be successful in the game). Examples are shown in figure 7 at different values of α, for $S \in \{3, 4\}$, both following unbiased initial conditions. The symmetry of the problem guarantees that for $N \to \infty$ the distribution $\varrho(\overline{v}_a)$ will be the same for all a (although it will clearly depend on S).

It is clear that the complexities of MGs with $S > 2$ are in the nontrivial dependence on control parameters and initial conditions of the frequencies with which the agents use their available strategies. Mathematically one finds this reflected in a nontrivial closure problem, as we will see below.

4. Generating functional analysis for general S

In solving the dynamics of the process (6), (7) for general S we can follow the strategy of the $S = 2$ case (see, e.g. [10, 11] or [4]), although we no longer benefit from simplifications such

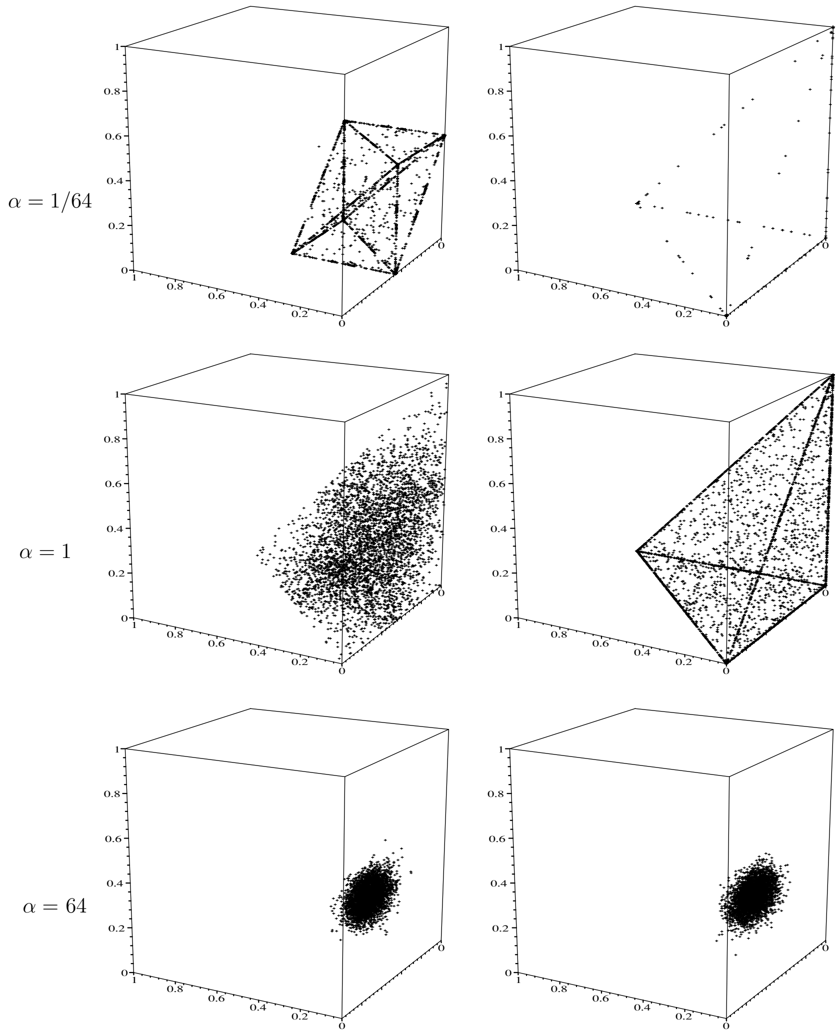

Figure 4. The plots show the first three components of the frequency vectors $\mathbf{f}_i = (f_1^i, f_2^i, f_3^i, f_4^i)$ with which the agents use their three strategies, drawn for each agent i as a point in $[0, 1]^3$ (giving N points per plot), obtained from numerical simulations of an MG without decision noise, with $N = 4097$ and $S = 4$. The constraint $\sum_a f_a^i = 1$ for all i now implies that all points are in a hyper-plane of which in the present graph one sees a three-dimensional projection. Left plots: unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$). Right plots: biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$).

as a reduction to equations and observables with valuation differences only. We will therefore suppress details and give only relevant intermediate stages of the calculation. For general S the canonical disorder-averaged[1] moment generating functional will be

$$\overline{Z[\psi]} = \overline{\left\langle \exp\left(\mathrm{i} \sum_{iat} \psi_{ia}(t) \delta_{a,a_i(t)} \right) \right\rangle}, \tag{8}$$

[1] As always, the pNS strategy entries R_μ^{ia} are regarded as frozen disorder, and disorder averages are written as $\overline{(\ldots)}$.

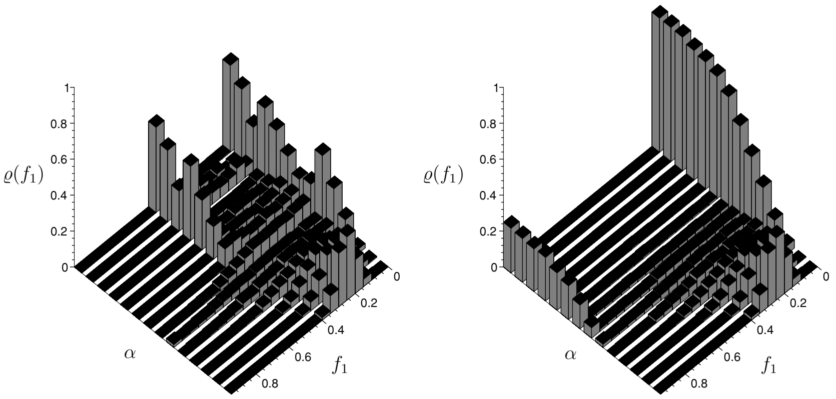

Figure 5. Histograms of the fraction $\varrho(f_1)$ of agents that play strategy 1 with frequency f in the stationary state (observed in simulations with $S = 4$), for $\alpha \in \{1/128, 1/64, \ldots, 32, 64\}$ (increasing by a factor of 2 at each step). Left: unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$). Right: biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$).

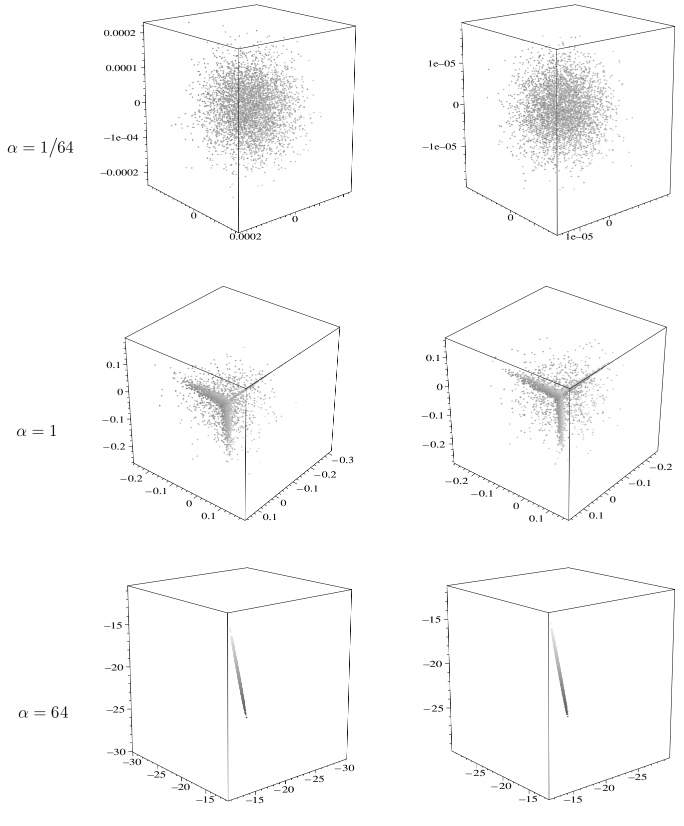

Figure 6. Each plot shows the *N* valuation velocity vectors $(\overline{v}_1^i, \overline{v}_2^i, \overline{v}_3^i)$ as points in $\mathbb{R}^3$, obtained from numerical simulations of an MG without decision noise, with $N = 4097$ and $S = 3$. Left plots: unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$). Right plots: biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$). Note the vastly different scales in the three graphs (increasing from top to bottom).

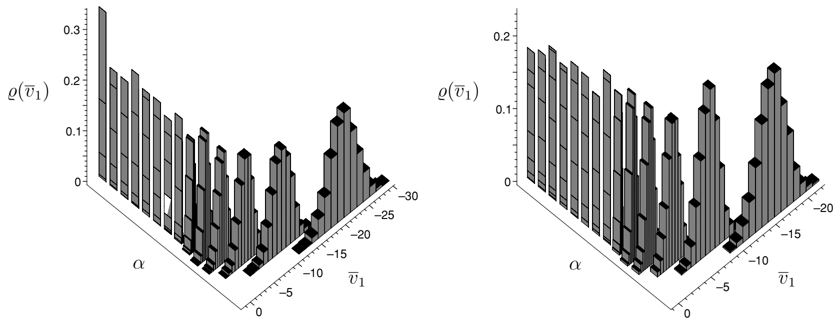

Figure 7. Histograms of the fraction $\varrho(\overline{v}_1)$ of agents that have strategy velocity $\overline{v}_1$ for strategy 1 in the stationary state (observed in simulations with $N = 4097$), for $\alpha \in \{1/128, 1/64, \ldots, 32, 64\}$ (increasing by a factor of 2 at each step). Left: $S = 3$. Right: $S = 4$. All measurements were taken following unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$).

where $a_i(t) = m(\mathbf{v}^i(t), \mathbf{z}^i(t))$ is the active strategy of agent i at time t. It generates dynamical observables such as the correlation and response functions:

$$C_{tt'} = \frac{1}{N}\sum_{ia} \overline{\langle \delta_{a,a_i(t)}\delta_{a,a_i(t')}\rangle} = -\lim_{\psi\to\mathbf{0}} \frac{1}{N}\sum_{ia} \frac{\partial^2 \overline{Z[\psi]}}{\partial\psi_{ia}(t)\partial\psi_{ia}(t')} \tag{9}$$

$$G_{tt'} = \frac{1}{N}\sum_{ia} \frac{\partial\overline{\langle \delta_{a,a_i(t)}\rangle}}{\partial\theta_{ia}(t')} = -\mathrm{i}\lim_{\psi\to\mathbf{0}} \frac{1}{N}\sum_{ia} \frac{\partial^2 \overline{Z[\psi]}}{\partial\psi_{ia}(t)\partial\theta_{ia}(t')}. \tag{10}$$

$C_{tt'}$ gives the likelihood that the active strategies at times t and t' are identical; $G_{tt'}$ measures the increase in probability of a strategy being active at time t following a perturbation of its valuation at time t' (both averaged over all agents and over the disorder). Causality guarantees that $G_{tt'} = 0$ for $t \leqslant t'$. We note that random strategy selection by the agents would give $C_{tt'} = S^{-1} + \delta_{tt'}(1 - S^{-1})$ and $G_{tt'} = 0$. A fully frozen state would be characterized by $C_{tt'} = 1$ and $G_{tt'} = 0$.

The functional (8) is an average over all possible paths of the combined N-agent state vector $\{\mathbf{v}\}$ in time. The probability density for each path is a product of the kernels $W_t(\ldots)$ in (7), so upon writing the δ-functions in (7) in integral form one finds

$$\begin{aligned}\overline{Z[\psi]} = \int \left[\prod_{it} \frac{\mathrm{d}\mathbf{v}^i(t)\,\mathrm{d}\hat{\mathbf{v}}^i(t)}{(2\pi)^S}\right] P_0(\mathbf{v}^1, \ldots, \mathbf{v}^N) \exp\left(\mathrm{i}\sum_{iat} \hat{v}^i_a(t)\big[v^i_a(t+1) - v^i_a(t) - \theta_{ia}(t)\big]\right) \\ \times \left\langle \exp\left(\mathrm{i}\sum_{iat} \psi_{ia}(t)\delta_{a,m(\mathbf{v}^i(t),\mathbf{z}^i(t))}\right)\right. \\ \left.\times \overline{\left[\exp\left(\mathrm{i}(\tilde{\eta}/N)\sum_{ijaa't\mu} R^{ia}_\mu R^{ja'}_\mu \hat{v}^i_a(t)\delta_{a',m(\mathbf{v}^j(t),\mathbf{z}^j(t))}\right)\right]}\right\rangle_{\{\mathbf{z}\}}.\end{aligned} \tag{11}$$

The dependence on strategy entries in the exponent is linearized by introducing auxiliary variables $x^\mu_t = \tilde{\eta}N^{-1/2}\sum_{ia}\hat{v}^i_a(t)R^{ia}_\mu$ (via suitable δ-functions), after which the disorder average is carried out. In leading order in N, the result depends on the $\{\mathbf{v}\}$ only via

$$C_{tt'}(\{\mathbf{v},\mathbf{z}\}) = \frac{1}{N}\sum_{ia} \delta_{a,m(\mathbf{v}^i(t),\mathbf{z}^i(t))}\delta_{a,m(\mathbf{v}^i(t'),\mathbf{z}^i(t'))} \tag{12}$$

$$K_{tt'}(\{\mathbf{v},\mathbf{z}\}) = \frac{1}{N}\sum_{ia}\delta_{a,m(\mathbf{v}^i(t),\mathbf{z}^i(t))}\hat{v}^i_a(t') \tag{13}$$

$$L_{tt'}(\{\mathbf{v},\mathbf{z}\}) = \frac{1}{N}\sum_{ia}\hat{v}^i_a(t)\hat{v}^i_a(t'). \tag{14}$$

Upon isolating (12)–(14) via integrals over suitable δ-functions we can factorize and integrate out the site-dependent variables, given site-factorized initial conditions. We then find an expression for $\overline{Z[\psi]}$ which can for $N\to\infty$ be evaluated by steepest descent:

$$\overline{Z[\psi]} = \int\left[\prod_{tt'}\mathrm{d}C_{tt'}\,\mathrm{d}\hat{C}_{tt'}\,\mathrm{d}K_{tt'}\,\mathrm{d}\hat{K}_{tt'}\,\mathrm{d}L_{tt'}\,\mathrm{d}\hat{L}_{tt'}\right]\exp(N(\Psi+\Phi+\Omega)+\mathcal{O}(\log N)) \tag{15}$$

$$\Psi = \mathrm{i}\sum_{tt'}[C_{tt'}\hat{C}_{tt'}+K_{tt'}\hat{K}_{tt'}+L_{tt'}\hat{L}_{tt'}] \tag{16}$$

$$\Phi = \alpha\log\int\left[\prod_t\frac{\mathrm{d}x_t\,\mathrm{d}\hat{x}_t}{2\pi}\,\mathrm{e}^{\mathrm{i}\hat{x}_t x_t}\right]\exp\left(-\frac{1}{2}\sum_{tt'}[x_tC_{tt'}x_{t'}+\tilde{\eta}^2\hat{x}_tL_{tt'}\hat{x}_{t'}-2\tilde{\eta}x_tK_{tt'}\hat{x}_{t'}]\right) \tag{17}$$

$$\begin{aligned}\Omega = \frac{1}{N}\sum_i\log\Bigg\langle&\left[\prod_{at}\frac{\mathrm{d}v_a(t)\,\mathrm{d}\hat{v}_a(t)}{2\pi}\right]P_0(\mathbf{v}(0))\\ &\times\exp\left(-\mathrm{i}\sum_{att'}[\hat{C}_{tt'}\delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}+\hat{L}_{tt'}\hat{v}_a(t)\hat{v}_a(t')+\hat{K}_{tt'}\delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\hat{v}_a(t')]\right)\\ &\times\exp\left(\mathrm{i}\sum_{at}\hat{v}_a(t)[v_a(t+1)-v_a(t)-\theta_{ia}(t)]+\mathrm{i}\sum_{at}\psi_{ia}(t)\delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\right)\Bigg\rangle.\end{aligned} \tag{18}$$

For $N\to\infty$ the order parameters $\{C,\hat{C},K,\hat{K},L,\hat{L}\}$ are determined by the saddle-point equations of the exponent $\Psi+\Phi+\Omega$. Working out these equations is straightforward but lengthy, see appendix A. The resulting theory can be written solely in terms of the kernels (9), (10), and, upon choosing $\theta_{ia}(t)=\theta_a(t)$, formulated in terms of the following stochastic process for the strategy valuations $\mathbf{v}=(v_1,\ldots,v_S)$ of a single 'effective agent', with zero-average coloured Gaussian noise forces $\{\eta_a\}$ and a retarded self-interaction:

$$v_a(t+1) = v_a(t)+\theta_a(t)-\alpha\sum_{t'\leqslant t}R_{tt'}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}+\sqrt{\alpha}\eta_a(t) \tag{19}$$

$$R_{tt'} = \tilde{\eta}(1\!\!1+\tilde{\eta}G)^{-1}_{tt'} \qquad \langle\eta_a(t)\eta_b(t')\rangle = \delta_{ab}(RCR^{\dagger})_{tt'}. \tag{20}$$

In terms of averages over the process (19), (20), the kernels C and G must be solved from

$$C_{tt'} = \sum_a\langle\delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}\rangle \qquad G_{tt'} = \sum_a\frac{\partial\langle\delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\rangle}{\partial\theta_a(t')}. \tag{21}$$

Finally, in a similar manner one derives exact expressions, in the limit $N\to\infty$, for the disorder-averaged bid covariance matrix $\overline{\Xi}_{tt'}$ and also for the volatility and predictability σ and H (which both should be self-averaging for $N\to\infty$). Here the appropriate generating functional is

$$\overline{Z[\phi]} = \overline{\left\langle\exp\left(\mathrm{i}\sum_{i\mu t}\phi_{i\mu}(t)A_\mu(t)\right)\right\rangle}. \tag{22}$$

From (22), which is calculated from the result of evaluating the previous functional $\overline{Z[\psi]}$ upon making simple substitutions (see appendix B for details), one obtains

$$\lim_{N\to\infty}\overline{\langle A_\mu(t)\rangle} = -\mathrm{i}\lim_{N\to\infty}\lim_{\psi\to\mathbf{0}}\frac{\partial\overline{Z[\phi]}}{\partial\phi_\mu(t)} = 0 \tag{23}$$

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{p}\sum_\mu\overline{\langle A_\mu(t)A_\mu(t')\rangle} &= -\lim_{N\to\infty}\frac{1}{p}\sum_\mu\lim_{\phi\to\mathbf{0}}\frac{\partial^2\overline{Z[\phi]}}{\partial\phi_\mu(t)\partial\phi_\mu(t')}\\ &= \tilde{\eta}^{-2}(RCR^\dagger)_{tt'}.\end{aligned} \tag{24}$$

It follows that in the limit $N\to\infty$ we must have $\overline{\Xi}_{tt'} = \frac{1}{\tilde{\eta}^2}(RCR^\dagger)_{tt'}$, and hence

$$\sigma^2 = \tilde{\eta}^{-2}\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}(RCR^\dagger)_{tt} \qquad H = \tilde{\eta}^{-2}\lim_{\tau\to\infty}\frac{1}{\tau^2}\sum_{tt'=1}^{\tau}(RCR^\dagger)_{tt'}. \tag{25}$$

5. Stationary states: the strategy frequency problem

To find time-translation invariant stationary solutions of our MG without anomalous response and with weak long-term memory we put $C_{tt'} = C(t-t')$ and $G_{tt'} = G(t-t')$, and we define the usual static order parameters, viz. the persistent correlations $c = \lim_{t\to\infty} C(t)$ and the static susceptibility $\chi = \sum_{t>0} G(t)$. Consequently also $R_{tt'} = R(t-t')$ and

$$\chi_{\mathrm{R}} = \sum_{t\geqslant 0} R(t) = \frac{\tilde{\eta}}{1+\tilde{\eta}\chi}. \tag{26}$$

We extract from (19), (20) an equation[2] for the valuation growth rates $\overline{v}_a = \lim_{t\to\infty} v_a(t)/t$. This involves the 'frozen' Gaussian fields $\overline{\eta}_a = \lim_{t\to\infty} t^{-1}\sum_{t'\leqslant t}\eta_a(t')$, the persistent perturbations $\overline{\theta}_a = \lim_{t\to\infty} t^{-1}\sum_{t'\leqslant t}\theta_a(t')$ and the strategy selection frequencies f_a:

$$\overline{v}_a = \overline{\theta}_a + \sqrt{\alpha}\,\overline{\eta}_a - \alpha\chi_{\mathrm{R}} f_a \qquad f_a = \lim_{t\to\infty}\frac{1}{t}\sum_{s\leqslant t}\langle\delta_{a,m(\mathbf{v}(s),\mathbf{z})}\rangle_{\mathbf{z}}. \tag{27}$$

Since $\langle\overline{\eta}_a\rangle = 0$ and $\langle\overline{\eta}_a\overline{\eta}_b\rangle = \delta_{ab}c\chi_{\mathrm{R}}^2$, we write $\overline{\eta}_a = \chi_{\mathrm{R}}\sqrt{c}x_a$, where $\mathbf{x} = (x_1,\ldots,x_S)$ is a vector of S uncorrelated zero-average and unit-variance frozen Gaussian variables. So (27) are $2S$ relations for the $2S$ unknown variables $(\overline{v}_1,\ldots,\overline{v}_S)$ and $(f_1,\ldots,f_S)$, parametrized by $\mathbf{x}$. Elimination of the valuation growth rates would give a formula for $f_a(\mathbf{x})$, with $a = 1,\ldots,S$. To emphasize this structure of our problem we write (27) as

$$f_a(\mathbf{x}) = x_a\sqrt{\frac{c}{\alpha}} + \frac{\overline{\theta}_a - \overline{v}_a(\mathbf{x})}{\alpha\chi_{\mathrm{R}}} \qquad f_a(\mathbf{x}) = \lim_{t\to\infty}\frac{1}{t}\sum_{s\leqslant t}\langle\delta_{a,m(\mathbf{v}(s,\mathbf{x}),\mathbf{z})}\rangle_{\mathbf{z}}. \tag{28}$$

One easily derives from (21) closed equations for the static order parameters, via $c = \lim_{t\to\infty} t^{-2}\sum_{ss'\leqslant t} C(s-s')$ and $\chi = \lim_{t\to\infty} t^{-1}\sum_{s\leqslant t}\partial\langle\delta_{a,m(\mathbf{v}(s),\mathbf{z}(s))}\rangle/\partial\overline{\theta}_a$, in terms of the solution $\{f_a(\mathbf{x})\}$ of (28). The equation for χ can be simplified further upon noting that $\partial/\partial\overline{\theta}_a = \alpha^{-1/2}\partial/\partial\overline{\eta}_a = (\chi_{\mathrm{R}}\sqrt{\alpha c})^{-1}\partial/\partial x_a$. The result is

$$c = \int D\mathbf{x}\sum_a f_a^2(\mathbf{x}) \qquad \chi = \frac{1}{\chi_{\mathrm{R}}\sqrt{\alpha c}}\int D\mathbf{x}\sum_a x_a f_a(\mathbf{x}) \tag{29}$$

(with the usual short-hand $Dx = (2\pi)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x$). The perturbation fields $\overline{\theta}_a$ are now no longer needed and can be set to zero, which simplifies (28) to

$$f_a(\mathbf{x}) = x_a\sqrt{\frac{c}{\alpha}} - \frac{\overline{v}_a(\mathbf{x})}{\alpha\chi_{\mathrm{R}}} \qquad f_a(\mathbf{x}) = \lim_{t\to\infty}\frac{1}{t}\sum_{s\leqslant t}\langle\delta_{a,m(\mathbf{v}(s,\mathbf{x}),\mathbf{z})}\rangle_{\mathbf{z}}. \tag{30}$$

[2] In the remainder of this paper we follow standard practice and use the notation $\overline{\cdots}$ for time averages. Since all disorder averages have at this stage been carried out, this cannot create ambiguities.

Solving the stationary state of the MG, including finding the phase transition marked by a divergence of χ, thus boils down to solving $\{f_a(\mathbf{x})\}$ from (30). This is the strategy frequency problem. The difficulty is in the second part of (30): even if two strategies (a, b) have $\overline{v}_a(\mathbf{x}) = \overline{v}_b(\mathbf{x})$ it does not follow that $f_a(\mathbf{x}) = f_b(\mathbf{x})$ (this is also clear in simulations). All valuation growth rates $\overline{v}_c(\mathbf{x})$, including those with $c \notin \{a, b\}$, will influence $f_a(\mathbf{x})$ and $f_b(\mathbf{x})$. The frequencies depend in a highly nontrivial way on the realization of the Gaussian vector $\mathbf{x}$ (which represents the diversity in the original N-agent population) and the control parameter α. Even the transients of the valuations, i.e. the full $v_a(s, \mathbf{x})$ rather than just the growth rates $\overline{v}_a(\mathbf{x})$, could in principle impact on the long-term frequencies $\{f_a(\mathbf{x})\}$. All this appears to make the problem practically insoluble. For $S = 2$ the situation could be saved upon translation of our equations into the language of valuation differences; there was only one relevant quantity, $\tilde{q} = \overline{v}_1 - \overline{v}_2$, and what mattered was only whether or not $\tilde{q} = 0$. For $S > 2$ this is no longer true.

6. Solution of the strategy frequency problem

We turn to the general solution of the strategy frequency problem, for arbitrary S and additive[3] decision noise: $m(\mathbf{v}, \mathbf{z}) = \operatorname{argmax}_a[v_a + Tz_a]$. This includes the deterministic case for $T = 0$. We define $\overline{v}^\star(\mathbf{x}) = \max_b \overline{v}_b(\mathbf{x})$, and the set $\Lambda(\mathbf{x})$ of all strategy indices for which $\overline{v}_a(\mathbf{x}) = \overline{v}^\star(\mathbf{x})$:

$$\Lambda(\mathbf{x}) = \left\{a|\overline{v}_a(\mathbf{x}) = \max_b \overline{v}_b(\mathbf{x})\right\} \subseteq \{1, \ldots, S\} \qquad |\Lambda(\mathbf{x})| > 0. \tag{31}$$

The solution now proceeds in three stages:

- Since $v_a(s, \mathbf{x}) = s\{\overline{v}_a(\mathbf{x}) + \varepsilon_a(s, \mathbf{x})\}$ with $\lim_{s\to\infty} \varepsilon_a(s, \mathbf{x}) = 0$ we may write for the second equation in (28)

$$f_a(\mathbf{x}) = \lim_{s\to\infty} \left\langle \prod_{b\neq a} \left\langle \theta\left[\overline{v}_a(\mathbf{x}) - \overline{v}_b(\mathbf{x}) + \varepsilon_a(s, \mathbf{x}) - \varepsilon_b(s, \mathbf{x}) + \frac{T(z - z')}{s}\right]\right\rangle_{z'} \right\rangle_z.$$

  This quantity can be nonzero only for $a \in \Lambda(\mathbf{x})$. Hence, once we know $\Lambda(\mathbf{x})$ and $\overline{v}^\star(\mathbf{x})$ the problem is solved, since in combination with (28) we may write

$$a \notin \Lambda(\mathbf{x}): \quad f_a(\mathbf{x}) = 0, \qquad \overline{v}_a(\mathbf{x}) = x_a \chi_{\mathrm{R}} \sqrt{\alpha c} \tag{32}$$

$$a \in \Lambda(\mathbf{x}): \quad f_a(\mathbf{x}) = x_a\sqrt{\frac{c}{\alpha}} - \frac{\overline{v}^\star(\mathbf{x})}{\alpha\chi_{\mathrm{R}}}, \qquad \overline{v}_a(\mathbf{x}) = \overline{v}^\star(\mathbf{x}). \tag{33}$$

- Next we calculate $\overline{v}^\star(\mathbf{x})$. Probability normalization guarantees that $\sum_a f_a(\mathbf{x}) = 1$ for any $\mathbf{x}$, but since $f_a(\mathbf{x}) \neq 0$ only for $a \in \Lambda(\mathbf{x})$ we have in fact $\sum_{a\in\Lambda(\mathbf{x})} f_a(\mathbf{x}) = 1$. Summing over the indices in (33) therefore leads to

$$\overline{v}^\star(\mathbf{x}) = \frac{\chi_{\mathrm{R}}\sqrt{\alpha c}}{|\Lambda(\mathbf{x})|} \sum_{a\in\Lambda(\mathbf{x})} x_a - \frac{\alpha\chi_{\mathrm{R}}}{|\Lambda(\mathbf{x})|}. \tag{34}$$

  Upon abbreviating $|\Lambda(\mathbf{x})|^{-1} \sum_{b\in\Lambda(\mathbf{x})} U_b = \langle U\rangle_{\Lambda(\mathbf{x})}$ our equations (32), (33) then become

$$a \notin \Lambda(\mathbf{x}): \quad f_a(\mathbf{x}) = 0, \qquad \overline{v}_a(\mathbf{x}) = x_a \chi_{\mathrm{R}} \sqrt{\alpha c} \tag{35}$$

$$a \in \Lambda(\mathbf{x}): \quad f_a(\mathbf{x}) = \frac{1}{|\Lambda(\mathbf{x})|} + \sqrt{\frac{c}{\alpha}}(x_a - \langle x\rangle_{\Lambda(\mathbf{x})}), \qquad \overline{v}_a(\mathbf{x}) = \overline{v}^\star(\mathbf{x}). \tag{36}$$

[3] The solution in the case of multiplicative decision noise is not identical but very similar to the one discussed here.

- What remains is to determine the set $\Lambda(\mathbf{x})$. The definition (31) of $\Lambda(\mathbf{x})$ demands that $\overline{v}_a(\mathbf{x}) < \overline{v}^\star(\mathbf{x})$ for all $a \notin \Lambda(\mathbf{x})$, i.e.

$$a \notin \Lambda(\mathbf{x}): \quad x_a < \langle x \rangle_{\Lambda(\mathbf{x})} - \sqrt{\frac{\alpha}{c}} \frac{1}{|\Lambda(\mathbf{x})|}. \tag{37}$$

However, we have similar inequalities for $a \in \Lambda(\mathbf{x})$, as (36) must obey $f_a(\mathbf{x}) \in [0, 1]$:

$$a \in \Lambda(\mathbf{x}): \quad x_a \geqslant \langle x \rangle_{\Lambda(\mathbf{x})} - \sqrt{\frac{\alpha}{c}} \frac{1}{|\Lambda(\mathbf{x})|} \tag{38}$$

$$x_a \leqslant \langle x \rangle_{\Lambda(\mathbf{x})} - \sqrt{\frac{\alpha}{c}} \frac{1}{|\Lambda(\mathbf{x})|} + \sqrt{\frac{\alpha}{c}}. \tag{39}$$

These last three groups of inequalities turn out to determine the set $\Lambda(\mathbf{x})$ uniquely. First, they tell us that $\Lambda(\mathbf{x})$ must contain the indices of the ℓ *largest* components of the vector $\mathbf{x}$, where $\ell = |\Lambda(\mathbf{x})|$. With probability one we may assume all components of $\mathbf{x}$ to be different, so for each $\mathbf{x}$ there is a unique permutation $\pi_{\mathbf{x}} : \{1, \ldots, S\} \to \{1, \ldots, S\}$ for which these components will be ordered according to $x_{\pi(1)} > x_{\pi(2)} > \cdots > x_{\pi(S)}$. We may now translate our key inequalities into the following three conditions, where $\ell = |\Lambda(\mathbf{x})| \in \{1, \ldots, S\}$ is the only quantity left to be solved, and with the ordering permutation $\pi = \pi_{\mathbf{x}}$:

$$x_{\pi(\ell)} \geqslant \frac{1}{\ell - 1} \sum_{m=1}^{\ell-1} x_{\pi(m)} - \frac{1}{\ell - 1}\sqrt{\frac{\alpha}{c}} \qquad \text{if} \quad \ell > 1 \tag{40}$$

$$x_{\pi(\ell+1)} < \frac{1}{\ell} \sum_{m=1}^{\ell} x_{\pi(m)} - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} \qquad \text{if} \quad \ell < S \tag{41}$$

$$x_{\pi(1)} \leqslant \frac{1}{\ell} \sum_{m=1}^{\ell} x_{\pi(m)} - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} + \sqrt{\frac{\alpha}{c}}. \tag{42}$$

The solution is the following: ℓ is the *smallest* number in $\{1, \ldots, S\}$ for which (41) holds (if any), whereas if (41) never holds then $\ell = S$. By construction we thereby satisfy both (40) and (41). What remains is to show that also (42) will be satisfied, and that the solution is unique. Clearly (42) holds when $\ell = 1$. To prove (42) for $\ell > 1$ we first define $X_k = k^{-1} \sum_{m \leqslant k} x_{\pi(m)}$ for $k \leqslant \ell$. We know from (40) that

$$\begin{aligned} X_k &= \frac{1}{k} \sum_{m=1}^{k-1} x_{\pi(m)} + \frac{1}{k} x_{\pi(k)} \\ &\geqslant \frac{1}{k} \sum_{m=1}^{k-1} x_{\pi(m)} + \frac{1}{k(k-1)} \sum_{m=1}^{k-1} x_{\pi(m)} - \frac{1}{k(k-1)}\sqrt{\frac{\alpha}{c}} = X_{k-1} - \frac{1}{k(k-1)}\sqrt{\frac{\alpha}{c}}. \end{aligned}$$

Hence $X_\ell \geqslant X_1 - \sqrt{\frac{\alpha}{c}} \sum_{m=2}^{\ell} \frac{1}{m(m-1)} = x_{\pi(1)} - \sqrt{\frac{\alpha}{c}} \sum_{m=2}^{\ell} \frac{1}{m(m-1)}$, so that

$$x_{\pi(1)} \leqslant \frac{1}{\ell} \sum_{m=1}^{\ell} x_{\pi(m)} + \sqrt{\frac{\alpha}{c}} \sum_{m=1}^{\ell-1} \frac{1}{m(m+1)} = \frac{1}{\ell} \sum_{m=1}^{\ell} x_{\pi(m)} + \sqrt{\frac{\alpha}{c}} \frac{\ell - 1}{\ell}.$$

This is the inequality (42) that we set out to prove. The corollary is that we have indeed defined a self-consistent solution of our equations. The above solution must be unique: any alternative choice of ℓ (rather than the smallest) that satisfies (41) would always make the previous (smallest) choice induce a violation of (40).

We summarize the solution of the strategy frequency problem for additive decision noise:

$$a \notin \Lambda(\mathbf{x}): \quad f_a(\mathbf{x}) = 0 \tag{43}$$

$$a \in \Lambda(\mathbf{x}): \quad f_a(\mathbf{x}) = \frac{1}{\ell(\mathbf{x})} + \sqrt{\frac{c}{\alpha}}(x_a - \langle x \rangle_{\Lambda(\mathbf{x})}) \tag{44}$$

$$\Lambda(\mathbf{x}) = \{\pi_{\mathbf{x}}(1), \ldots, \pi_{\mathbf{x}}(\ell(\mathbf{x}))\} \tag{45}$$

$$\pi_{\mathbf{x}}: \quad \text{permutation such that } x_{\pi(1)} > x_{\pi(2)} > \cdots > x_{\pi(S)} \tag{46}$$

$$\ell(\mathbf{x}): \quad \text{defined by the conditions} \tag{47}$$

$$x_{\pi(\ell+1)} < \frac{1}{\ell}\sum_{m=1}^{\ell} x_{\pi(m)} - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} \qquad \text{if} \quad \ell < S \tag{48}$$

$$x_{\pi(k+1)} > \frac{1}{k}\sum_{m=1}^{k} x_{\pi(m)} - \frac{1}{k}\sqrt{\frac{\alpha}{c}} \qquad \text{for all} \quad k < \ell. \tag{49}$$

The meaning of this solution is as follows. The randomness induced by the Gaussian variables $\mathbf{x}$ represents the variability in the original N-agent population. The set $\Lambda(\mathbf{x})$ contains the strategies that will be played by the effective agent, albeit with different frequencies. Strategies $a \notin \Lambda(\mathbf{x})$ are never played. Apparently, for a strategy a to be in the 'active' set $\Lambda(\mathbf{x})$, it must have an x_a that is sufficiently large, and sufficiently close to the components of the other strategies in the active set.

Since expressions (43), (44) depend only on $\Lambda(\mathbf{x})$, i.e. on $\pi_{\mathbf{x}}$ and ℓ (see above), we may abbreviate these formulae as $f_a(\mathbf{x}|\pi_{\mathbf{x}}, \ell)$. Averages of the form $\int D\mathbf{x}\, \Phi(\{x_a, f_a(\mathbf{x})\})$ can now be written as a sum over all permutations π of $\{1, \ldots, S\}$, with a function $C(\pi|\mathbf{x}) = \delta_{\pi,\pi_{\mathbf{x}}}$ that selects the right component ordering permutation for each $\mathbf{x}$:

$$\begin{aligned}
\int D\mathbf{x}\, \Phi(\{x_a, f_a(\mathbf{x})\}) &= \sum_{\pi}\sum_{\ell=1}^{S}\int D\mathbf{x}\, C(\pi|\mathbf{x})\Phi(\{x_a, f_a(\mathbf{x}|\pi, \ell)\}) \\
&\quad \times \prod_{a<\ell}\theta\left[x_{\pi(a+1)} - \frac{1}{a}\sum_{m=1}^{a} x_{\pi(m)} + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right] \\
&\quad \times \left\{\delta_{\ell S} + (1-\delta_{\ell S})\theta\left[\frac{1}{\ell}\sum_{m=1}^{\ell} x_{\pi(m)} - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} - x_{\pi(\ell+1)}\right]\right\},
\end{aligned} \tag{50}$$

where

$$C(\pi|\mathbf{x}) = \prod_{a=1}^{S-1}\theta[x_{\pi(a)} - x_{\pi(a+1)}]. \tag{51}$$

7. The static theory for arbitrary S

In those cases where one seeks to calculate the average of a function Φ that is *invariant* under all permutations of the index set $\{1, \ldots, S\}$, as in (29), one may use this invariance to simplify the calculation. Here the average will equal the contribution from one particular ordering of the components of $\mathbf{x}$ (and its associated permutation $\pi_{\mathbf{x}}$, for which we may take the identity permutation) times the number $S!$ of permutations:

$$\int D\mathbf{x}\,\Phi(\{x_a, f_a(\mathbf{x})\}) = S!\sum_{\ell=1}^{S}\int D\mathbf{x}\prod_{a=1}^{S-1}\theta[x_a - x_{a+1}]\Phi(\{x_a, f_a(\mathbf{x}|\ell)\})$$
$$\times\prod_{a<\ell}\theta\left[x_{a+1} - \frac{1}{a}\sum_{m=1}^{a}x_m + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right]$$
$$\times\left\{\delta_{\ell S} + (1-\delta_{\ell S})\theta\left[\frac{1}{\ell}\sum_{m=1}^{\ell}x_m - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} - x_{\ell+1}\right]\right\}. \tag{52}$$

In particular, upon application to (29) and using (43), (44) we have now arrived at fully explicit and closed equations for our static order parameters:

$$c = S!\sum_{\ell=1}^{S}\int D\mathbf{x}\prod_{a=1}^{S-1}\theta[x_a - x_{a+1}]\prod_{a<\ell}\theta\left[x_{a+1} - \frac{1}{a}\sum_{m=1}^{a}x_m + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right]$$
$$\times\left\{\delta_{\ell S} + (1-\delta_{\ell S})\theta\left[\frac{1}{\ell}\sum_{m=1}^{\ell}x_m - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} - x_{\ell+1}\right]\right\}\sum_{a=1}^{\ell}\left[\frac{1}{\ell} + \sqrt{\frac{c}{\alpha}}\left(x_a - \frac{1}{\ell}\sum_{m=1}^{\ell}x_m\right)\right]^2 \tag{53}$$

$$\frac{\tilde{\eta}\chi}{1+\tilde{\eta}\chi} = \frac{S!}{\sqrt{\alpha c}}\sum_{\ell=1}^{S}\int D\mathbf{x}\prod_{a=1}^{S-1}\theta[x_a - x_{a+1}]\prod_{a<\ell}\theta\left[x_{a+1} - \frac{1}{a}\sum_{m=1}^{a}x_m + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right]$$
$$\times\left\{\delta_{\ell S} + (1-\delta_{\ell S})\theta\left[\frac{1}{\ell}\sum_{m=1}^{\ell}x_m - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} - x_{\ell+1}\right]\right\}$$
$$\times\sum_{a=1}^{\ell}x_a\left[\frac{1}{\ell} + \sqrt{\frac{c}{\alpha}}\left(x_a - \frac{1}{\ell}\sum_{m=1}^{\ell}x_m\right)\right]. \tag{54}$$

For any given value of α one first solves (53) for c, after which χ is calculated via (54).

For arbitrary S we must generalize the concept of 'frozen agents': we define ϕ_k as the fraction of agents that in the stationary state play k of their S strategies. Since k is the size of the active set $\Lambda(\mathbf{x})$ in the language of (52) the order parameter ϕ_k is the average of the function $\Phi(\{x_a, f_a(\mathbf{x}|\ell)\}) = \delta_{k\ell}$, so our solution immediately tells us that

$$\phi_{\ell<S} = S!\int D\mathbf{x}\prod_{a=1}^{S-1}\theta[x_a - x_{a+1}]\prod_{a=1}^{\ell}\theta\left[x_a - \frac{1}{a}\sum_{m=1}^{a}x_m + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right]$$
$$\times\theta\left[\frac{1}{\ell}\sum_{m=1}^{\ell}x_m - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} - x_{\ell+1}\right] \tag{55}$$

$$\phi_S = S!\int D\mathbf{x}\prod_{a=1}^{S-1}\theta[x_a - x_{a+1}]\prod_{a=1}^{S}\theta\left[x_a - \frac{1}{a}\sum_{m=1}^{a}x_m + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right]. \tag{56}$$

These expressions obey $\sum_{k=1}^{S}\phi_k = 1$, as they should. Formulae (55), (56) can be simplified, by exploiting permutation-invariant parts. For instance, ϕ_1 and ϕ_2 can be rewritten as

$$\phi_1 = S!\int D\mathbf{x}\,\theta\left[x_1 - x_2 - \sqrt{\frac{\alpha}{c}}\right]\prod_{m=3}^{S}\theta[x_{m-1} - x_m]$$
$$= S\int \mathrm{D}x\left\{\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}} - \sqrt{\frac{\alpha}{2c}}\right)\right\}^{S-1} \tag{57}$$

$$\phi_2 = S! \int D\mathbf{x}\, \theta\left[x_2 - x_1 + \sqrt{\frac{\alpha}{c}}\right] \theta\left[x_1 + x_2 - 2x_3 - \sqrt{\frac{\alpha}{c}}\right] \prod_{m=2}^{S} \theta[x_{m-1} - x_m]$$
$$= S(S-1) \int \mathrm{D}x \int_0^{\sqrt{\alpha/c}} \frac{\mathrm{d}u}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}(x-u)^2} \left\{\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{2x-u}{2\sqrt{2}} - \frac{1}{2}\sqrt{\frac{\alpha}{2c}}\right)\right\}^{S-2}. \tag{58}$$

We can now also calculate the disorder-averaged strategy frequency distribution $\varrho(f) = \lim_{N\to\infty} N^{-1} \sum_i \overline{\langle \delta[f - f_{ai}]\rangle}$, the fraction of agents in the stationary state that use strategy a with frequency f. The problem is strategy permutation invariant, so $\varrho(f)$ cannot depend on a. We may therefore write it in the permutation-invariant form $\varrho(f) = \lim_{N\to\infty} (SN)^{-1} \sum_{ia} \overline{\langle \delta[f - f_{ai}]\rangle}$, and calculate it by applying formula (52) to the function $\Phi(\{x_a, f_a(\mathbf{x}|\ell)\}) = S^{-1} \sum_{a=1}^{S} \delta[f - f_a(\mathbf{x}|\ell)]$. This gives

$$\varrho(f) = \delta(f) \sum_{\ell=1}^{S-1} \left(1 - \frac{\ell}{S}\right) \phi_\ell + \delta(f-1)\frac{1}{S}\phi_1$$
$$+ (S-1)! \sum_{\ell=2}^{S} \sum_{b=1}^{\ell} \int D\mathbf{x}\, \delta\left[f - \frac{1}{\ell} - \sqrt{\frac{c}{\alpha}}\left(x_b - \frac{1}{\ell}\sum_{m=1}^{\ell} x_m\right)\right]$$
$$\times \prod_{a=1}^{S-1} \theta[x_a - x_{a+1}] \prod_{a=1}^{\ell-1} \theta\left[x_{a+1} - \frac{1}{a}\sum_{m=1}^{a} x_m + \frac{1}{a}\sqrt{\frac{\alpha}{c}}\right]$$
$$\times \left\{\delta_{\ell S} + (1 - \delta_{\ell S})\theta\left[\frac{1}{\ell}\sum_{m=1}^{\ell} x_m - \frac{1}{\ell}\sqrt{\frac{\alpha}{c}} - x_{\ell+1}\right]\right\}. \tag{59}$$

One can recover the full generating functional theory of the $S = 2$ batch MG as in e.g. [10, 11] from the above more general equations, as a test (taking into account carefully the different definitions of correlation and response functions that were made in earlier studies). We will not give details of this in principle straightforward exercise here.

In the limit $\alpha \to \infty$ our theory predicts that the MG will behave for any S as if the agents were to select strategies completely randomly, as one expects. Here, upon using $S! \sum_{a=1}^{S} \int D\mathbf{x} \prod_{a=1}^{S-1} \theta[x_a - x_{a+1}] = 1$ one finds that our equations simplify to

$$\lim_{\alpha\to\infty} c = S^{-1} \qquad \lim_{\alpha\to\infty} \chi = 0 \tag{60}$$
$$\lim_{\alpha\to\infty} \phi_\ell = \delta_{\ell S} \qquad \lim_{\alpha\to\infty} \varrho(f) = \delta\left(f - \frac{1}{S}\right). \tag{61}$$

Finally, the $\chi = \infty$ phase transition in the MG occurs when the right-hand side of (54) equals one, and defines the critical point $\alpha_c(S)$. Numerical solution of (53), (54) for $S \in \{2, 3, 4, 5\}$ gives the following values (accurate up to the last digit given):

S	2	3	4	5
$\alpha_c(S)$	0.337	0.824	1.324	1.822

These values are remarkably close to those one would predict on the basis of the heuristic relation $\alpha(S) \approx \alpha_c(2) + \frac{1}{2}(S-2)$, as proposed in [6], but not identical. Upon assuming the phase transition to be of a dimensional nature, the authors of [6] also derived an exact equation from which to solve $\alpha_c(S)$ (which at that stage could not be closed, except for $S = 2$ where it is indeed correct), which in the present notation reads

$$\sum_{\ell=1}^{S} \phi_\ell \ell = \alpha + 1. \tag{62}$$

We have not yet been able[4] to confirm the validity of (62) for general S. For $S = 3$, however, equation (62) reduces to $\phi_2 + 2\phi_3 = \alpha$, which one can easily confirm to be correct from the more explicit formulae derived in the next section; insertion of (68), (69) into $\phi_2 + 2\phi_3 = \alpha$ immediately reproduces condition (65) that marks the transition.

Below the critical point, in the nonergodic regime $\alpha < \alpha_c(S)$ the above theory no longer applies, as its assumption of finite integrated response χ is violated. For $S = 2$ it was shown [10, 11] that upon replacing (53) by the equation $\chi^{-1} = 0$, where χ is calculated from (54), the agreement between ergodic theory and simulations with biased initializations in the nonergodic regime could be improved. For $S > 2$ such heuristic improvements are again possible but more awkward. Furthermore, they are entirely ad hoc and artificial, and therefore mostly of cosmetic merit, so we have decided to restrict ourselves here to the theory (53), (54).

8. Application to MGs with $S = 3$

The case $S = 3$ is the simplest situation where all the complexities of having more than two strategies per agent can be studied, so we deal with this in detail. Here, with persistence, one can do most of the nested integrations in (53)–(56), (59) analytically. We give some of the basic identities involved in appendix C. The result is most easily expressed in parametric form in terms of the auxiliary variable $u = \sqrt{\alpha/c}$. The order parameter equation (53) for the persistent correlations c then becomes

$$c = 1 + \left(1 - \frac{3c}{\alpha}\right) I(u) - \frac{1}{4}\,\mathrm{erf}\left(\frac{u}{2}\right)\left[3 + \mathrm{erf}\left(\frac{u}{2\sqrt{3}}\right)\right] + \frac{3}{2u^2}\left[\mathrm{erf}\left(\frac{u}{2}\right) - \frac{u}{\sqrt{\pi}}\mathrm{e}^{-\frac{1}{2}u^2}\right]$$
$$+ \frac{3}{2u^2}\,\mathrm{erf}\left(\frac{u}{2}\right)\left[\mathrm{erf}\left(\frac{u}{2\sqrt{3}}\right) - \frac{u}{\sqrt{3\pi}}\mathrm{e}^{-\frac{1}{12}u^2}\right] + \frac{3}{2u\sqrt{\pi}}\,\mathrm{erf}\left(\frac{u}{2\sqrt{3}}\right)\mathrm{e}^{-\frac{1}{4}u^2} \tag{63}$$

with $I(u) = 2\int_0^{u/\sqrt{2}} \mathrm{D}x\,\mathrm{erf}(x/\sqrt{6})$ (this latter integral we could unfortunately not do analytically, except for the limit $I(\infty) = 1/3$). Solving equation (63) for c gives c as a function of α; the result is shown and tested against simulation data in figure 8 (left panel). Similarly, equation (54) for the susceptibility χ takes the explicit form

$$\frac{\tilde{\eta}\chi}{1 + \tilde{\eta}\chi} = \frac{3}{2\alpha}\left\{\mathrm{erf}\left(\frac{u}{2}\right)\left[1 + \mathrm{erf}\left(\frac{u}{2\sqrt{3}}\right)\right] - 2I(u)\right\}. \tag{64}$$

It follows that the $\chi = \infty$ phase transition occurs at a value α_c that must be solved (numerically) from the following two coupled equations:

$$\alpha = \frac{3}{2}\left\{\mathrm{erf}\left(\frac{u}{2}\right)\left[1 + \mathrm{erf}\left(\frac{u}{2\sqrt{3}}\right)\right] - 2I(u)\right\} \tag{65}$$

[4] In principle it should be possible to confirm (62) and study the behaviour of $\alpha_c(S)$ for arbitrary S from our order parameter equations. In practice, however, the proliferation with increasing S of nested integrations in these equations limits the feasibility of such analyses to modest values of S.

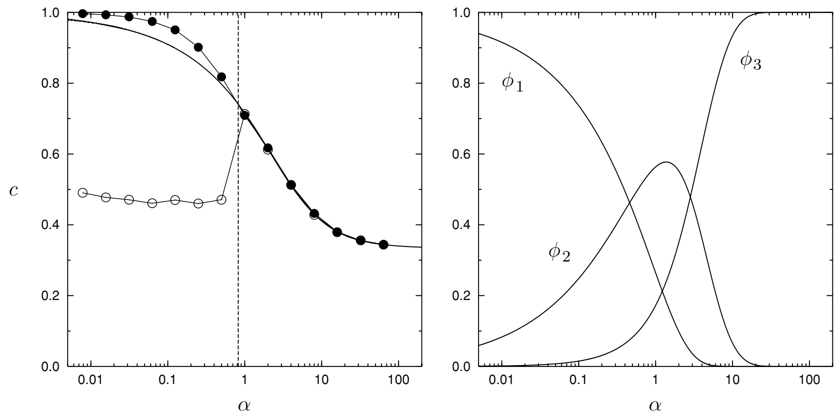

Figure 8. Left: the persistent correlations $c = \lim_{\tau\to\infty} C(\tau)$ (solid curve) for $S = 3$, as a function of $\alpha = p/N$, calculated for ergodic stationary states. They are shown together with data measured in numerical simulations of MGs without decision noise, for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, •) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, ○). Simulation system size: $N = 4097$. Dashed: the phase transition point $\alpha_c(3) \approx 0.824$. Right: the fractions ϕ_ℓ of agents that play precisely $\ell \in \{1, 2, 3\}$ of their $S = 3$ strategies, as a function of $\alpha = p/N$, calculated for ergodic stationary states. Note that $\phi_1 + \phi_2 + \phi_3 = 1$.

$$\frac{\alpha}{u^2} = 1 + \left(1 - \frac{3c}{\alpha}\right) I(u) - \frac{1}{4}\operatorname{erf}\left(\frac{u}{2}\right)\left[3 + \operatorname{erf}\left(\frac{u}{2\sqrt{3}}\right)\right] + \frac{3}{2u^2}\left[\operatorname{erf}\left(\frac{u}{2}\right) - \frac{u}{\sqrt{\pi}}\mathrm{e}^{-\frac{1}{2}u^2}\right]$$
$$+ \frac{3}{2u^2}\operatorname{erf}\left(\frac{u}{2}\right)\left[\operatorname{erf}\left(\frac{u}{2\sqrt{3}}\right) - \frac{u}{\sqrt{3\pi}}\mathrm{e}^{-\frac{1}{12}u^2}\right] + \frac{3}{2u\sqrt{\pi}}\operatorname{erf}\left(\frac{u}{2\sqrt{3}}\right)\mathrm{e}^{-\frac{1}{4}u^2}. \tag{66}$$

Upon solving these equations one finds that $\alpha_c \approx 0.824$. After doing the integrals in our expressions (55), (56) for the fractions ϕ_ℓ of agents that play ℓ of their three strategies one obtains the following explicit formulae (expressed once more in terms of $u = \sqrt{\alpha/c}$, i.e. in terms of the solution of our previous equation for c), which indeed obey $\phi_1 + \phi_2 + \phi_3 = 1$:

$$\phi_1 = 1 - \frac{3}{2}\operatorname{erf}\left(\frac{u}{2}\right) + \frac{3}{2}I(u) \tag{67}$$

$$\phi_2 = \frac{3}{2}\operatorname{erf}\left(\frac{u}{2}\right)\left[1 - \operatorname{erf}\left(\frac{u}{2\sqrt{3}}\right)\right] \tag{68}$$

$$\phi_3 = \frac{3}{2}\operatorname{erf}\left(\frac{u}{2}\right)\operatorname{erf}\left(\frac{u}{2\sqrt{3}}\right) - \frac{3}{2}I(u). \tag{69}$$

These three expressions are shown as functions of α in figure 8, and tested against numerical simulations in figure 9. For large α the agents tend to use all three strategies, upon reducing α one finds increasing numbers switching to the use of only one or only two of their strategies. Again, as in the figures involving c, we observe a perfect agreement between theory and simulations in the regime where our theory applies, i.e. for $\alpha > \alpha_c$. Furthermore, the value found for α_c is consistent with the simulation data in that non-ergodicity (a dependence on whether one chooses biased or unbiased strategy valuations at time zero) is indeed seen to set in at the predicted point.

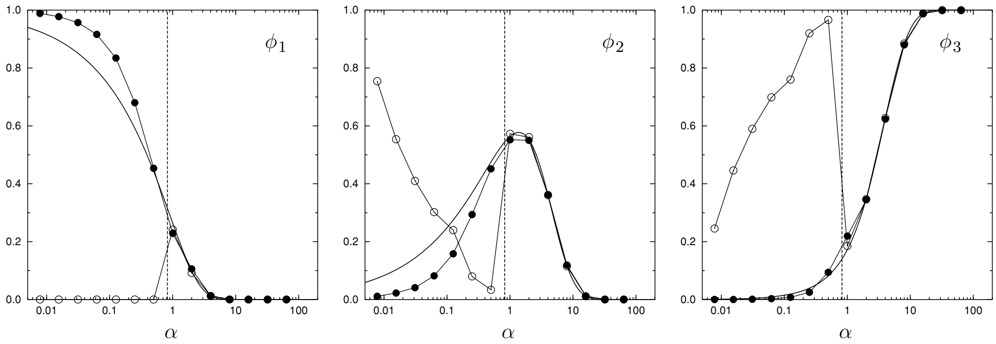

Figure 9. The fractions ϕ_ℓ for $S = 3$ are shown together with data measured in simulations of MGs without decision noise, for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, $\bullet$) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, $\circ$). Simulation system size: $N = 4097$. Dashed: the phase transition point $\alpha_c(3) \approx 0.824$.

Perhaps the most sensitive test of our $S = 3$ theory is to work out and validate formula (59) for the strategy frequency distribution. Upon doing the integrals we find confirmed that $\varrho(f) = 0$ for $f \notin [0, 1]$ (as it should), whereas for $f \in [0, 1]$ we arrive at

$$\varrho(f) = \frac{u}{\sqrt{\pi}}\,\mathrm{e}^{-u^2(f-\frac{1}{2})^2}\left[1 - \mathrm{erf}\left(\frac{u}{2\sqrt{3}}\right)\right] + \frac{u\sqrt{3}}{2\sqrt{\pi}}\,\mathrm{e}^{-\frac{3}{4}u^2(f-\frac{1}{3})^2}\,\mathrm{erf}\left(\frac{u}{2}(1-f)\right)$$
$$+\,\delta(f)\left(\frac{2}{3}\phi_1 + \frac{1}{3}\phi_2\right) + \delta(f-1)\frac{1}{3}\phi_1. \qquad (70)$$

One can understand this formula qualitatively. A strategy a is not played at all when either $\ell = 1$ and it is among the two non-selected strategies (this happens with probability $\frac{2}{3}\phi_1$), or when $\ell = 2$ and it is the one non-selected strategy (this happens with probability $\frac{1}{3}\phi_2$). A strategy a is played permanently if $\ell = 1$ and it is the selected one (this happens with probability $\frac{1}{3}\phi_1$). Hence the second line of (70). The first line of (70) reflects a being played now and then (among other strategies), with non-trivial frequencies; here one observes the expected local maxima at $f = \frac{1}{2}$ (corresponding to the 'average' behaviour for $\ell = 2$) and at $f = \frac{1}{3}$ (corresponding to the 'average' behaviour for $\ell = 3$). Also the prediction (70) agrees perfectly with numerical simulations, as shown in figure 10, in the regime $\alpha > \alpha_c$ for which it is supposed to be correct.

In summary, we may conclude that for $S = 3$ all our theoretical predictions regarding stationary state order parameters, the location of the phase transition, and even quantities such as the strategy frequency distribution, make physical sense and find perfect confirmation in numerical simulations.

9. Application to MGs with $S = 4, 5$

Although it is in principle still possible to proceed with the integrals in (53)–(56), (59) analytically, for $S > 3$ this becomes a very tedious exercise. We have here resorted to numerical evaluation; one does not expect qualitatively different physics to emerge compared to $S = 3$, and life is short. For reasons of brevity we have also restricted ourselves to the validation of the static observables c and ϕ_ℓ only. It should be mentioned that upon increasing the value of S, the accurate numerical evaluation (based on the Gauss–Legendre method) of

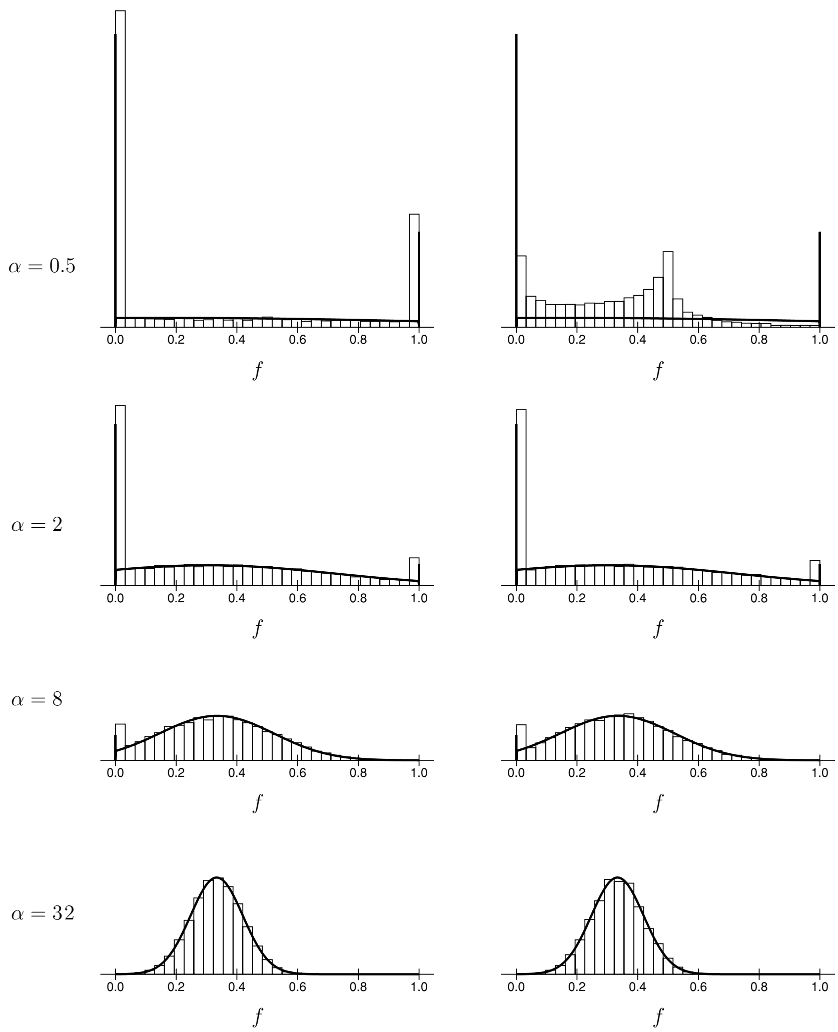

Figure 10. The strategy frequency distribution $\varrho(f)$ (thick curves), for $S = 3$ and different values of $\alpha = p/N$, as calculated for ergodic stationary states. They are shown together with strategy frequency data measured in numerical simulations of MGs without decision noise (histograms, averaged over three samples), for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, left graphs) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, right graphs). Note: $\alpha_c(3) \approx 0.824$, so the top two graphs refer to the nonergodic regime, where the present theory is not supposed to apply.

the various nested integrals in our theoretical expressions becomes nontrivial in terms of CPU quite quickly; especially for small values of α. Finding the expressions for $\alpha_c(S)$ as presented in section 7 for $S > 3$ already involved a careful assessment of the scaling of these values with the parameters that control the numerical integration accuracy. One also finds that for increasing values of S the finite size effects in the non-ergodic regime $\alpha < \alpha_c(S)$ become more prominent.

For $S = 4$ the results of evaluating numerically the theoretical predictions (53)–(56) are shown and tested against simulation data (obtained for MGs without decision noise) in

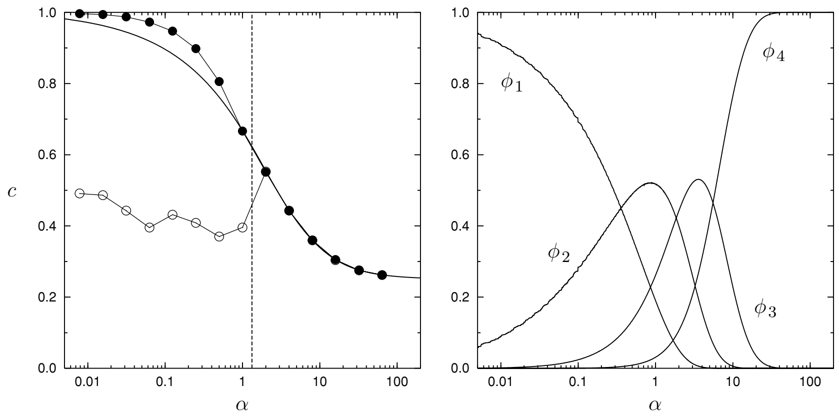

Figure 11. Left: the persistent correlations $c = \lim_{\tau\to\infty} C(\tau)$ (solid curve) for $S = 4$, as a function of $\alpha = p/N$, calculated for ergodic stationary states. They are shown together with corresponding data measured in numerical simulations of MGs without decision noise, for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, $\bullet$) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, $\circ$). Simulation system size: $N = 4097$. Dashed: the phase transition point $\alpha_c(4) \approx 1.324$. Right: the fractions ϕ_ℓ of agents that play precisely $\ell \in \{1, 2, 3, 4\}$ of their $S = 4$ strategies, as a function of $\alpha = p/N$, calculated for ergodic stationary states. Note that $\phi_1 + \phi_2 + \phi_3 + \phi_4 = 1$.

figures 11 and 12. For $S = 5$ they are shown and tested in figures 13 and 14. It is very satisfactory that once more we observe for $S \in \{4, 5\}$ in all cases an excellent agreement between theory and simulation data, both with regards to the observables measured (viz. c and the fractions ϕ_ℓ), and in terms of the locations of the transition points $\alpha_c(4)$ and $\alpha_c(5)$.

10. Stationary state fluctuations: volatility and predictability

Once the static order parameters c and χ are known, our formulae (25) for the volatility σ and the predictability measure H can be reduced to

$$\sigma^2 = \tilde{\eta}^{-2} \sum_{tt'\geqslant 0} R(t)C(t-t')R(t') \qquad H = \frac{c}{(1+\tilde{\eta}\chi)^2}. \tag{71}$$

As for $S = 2$, H can always be expressed fully in terms of persistent order parameters and vanishes at the phase transition point. The result is shown in figure 15 for $S \in \{3, 4, 5\}$. If in the volatility formula we separate the correlation function into a static and a non-persistent part, $C(t) = c + \tilde{C}(t)$ with $\lim_{t\to\pm\infty} \tilde{C}(t) = 0$, we obtain

$$\begin{aligned}\sigma^2 &= \frac{c}{(1+\tilde{\eta}\chi)^2} + \sum_{tt'\geqslant 0} (\mathbb{1}+\tilde{\eta}G)^{-1}(t)\tilde{C}(t-t')(\mathbb{1}+\tilde{\eta}G)^{-1}(t') \\ &= \frac{c}{(1+\tilde{\eta}\chi)^2} + (1-c)\sum_t [(\mathbb{1}+\tilde{\eta}G)^{-1}(t)]^2 \\ &\quad + \sum_{t\neq t'} (\mathbb{1}+\tilde{\eta}G)^{-1}(t)\tilde{C}(t-t')(\mathbb{1}+\tilde{\eta}G)^{-1}(t'). \end{aligned} \tag{72}$$

Expression (72) is still exact. The first term is recognized to be H. The other terms contain the short-time fluctuations, involving non-persistent dynamic order parameters. If one seeks a formula for σ in terms static order parameters only, one must pay the price of approximation. To do this we generalize the two procedures described for the $S = 2$ batch MG in e.g.

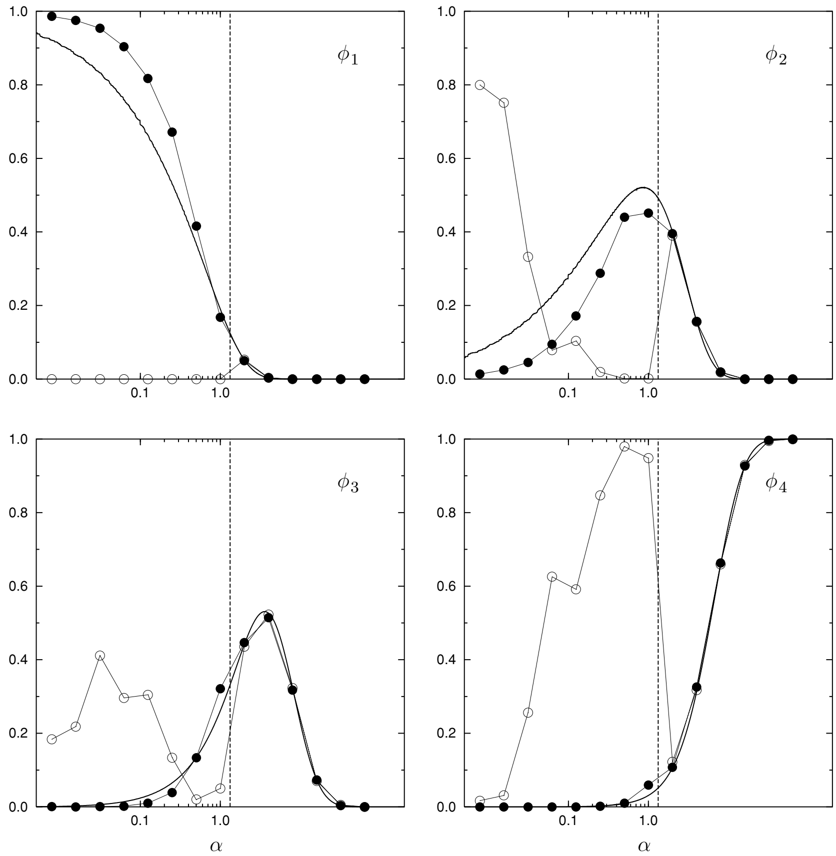

Figure 12. The fractions ϕ_ℓ for $S = 4$ are shown together with corresponding data measured in numerical simulations of MGs without decision noise, for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, $\bullet$) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, $\circ$). Dashed: the phase transition point $\alpha_c(4) \approx 1.324$.

[10, 11] and [4]. Both rely on the ansatz, motivated by observations in simulations, that the response function decays to zero very slowly, subject to the constraint $\sum_t G(t) = \chi$. Upon putting e.g., $G(t > 0) = \chi(\mathrm{e}^z - 1)\mathrm{e}^{-zt}$ with $z \to 0$ one finds for such kernels that $\lim_{z\to 0} \sum_t [(\mathbb{1} + \tilde{\eta}G)(t)]^2 = 0$. Consequently we may write

$$\sigma^2 \approx \frac{c}{(1+\tilde{\eta}\chi)^2} + \sum_{t\neq t'} (\mathbb{1} + \tilde{\eta}G)^{-1}(t)\tilde{C}(t-t')(\mathbb{1} + \tilde{\eta}G)^{-1}(t'). \tag{73}$$

The most brutal approximation of $\tilde{C}$ is to assume the correlations to decay to zero very fast, and simply put $\tilde{C}(t) \to \tilde{C}(0)\delta_{t0} = (1-c)\delta_{tt'}$. This leads to the formula

$$\sigma_A^2 = \frac{c}{(1+\tilde{\eta}\chi)^2} + 1 - c. \tag{74}$$

Alternatively, we may write the average in the definition (21) of the correlation function as a sum of contributions representing the possible sizes ℓ of the set of active strategies:

$$C(t-t') = \delta_{tt'} + (1-\delta_{tt'}) \sum_{\ell=1}^{S} \phi_\ell \sum_a \langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t)}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t')} \rangle_{\ell\,\text{active}}. \tag{75}$$

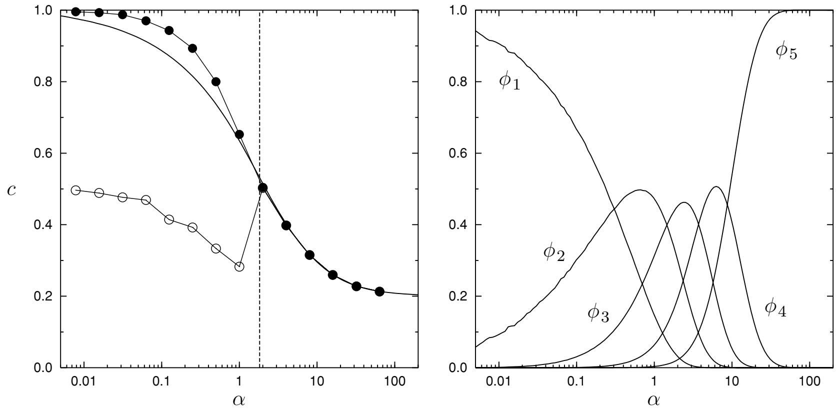

Figure 13. Left: persistent correlations $c = \lim_{\tau\to\infty} C(\tau)$ (solid curve) for $S = 5$, as a function of $\alpha = p/N$, calculated for ergodic stationary states. It is shown together with corresponding data measured in numerical simulations of MGs without decision noise, for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, $\bullet$) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, $\circ$). Dashed: the phase transition point $\alpha_c(5) \approx 1.822$. Right: the fractions ϕ_ℓ of agents that play precisely $\ell \in \{1, 2, 3, 4, 5\}$ of their $S = 5$ strategies, as a function of $\alpha = p/N$, calculated for ergodic stationary states.

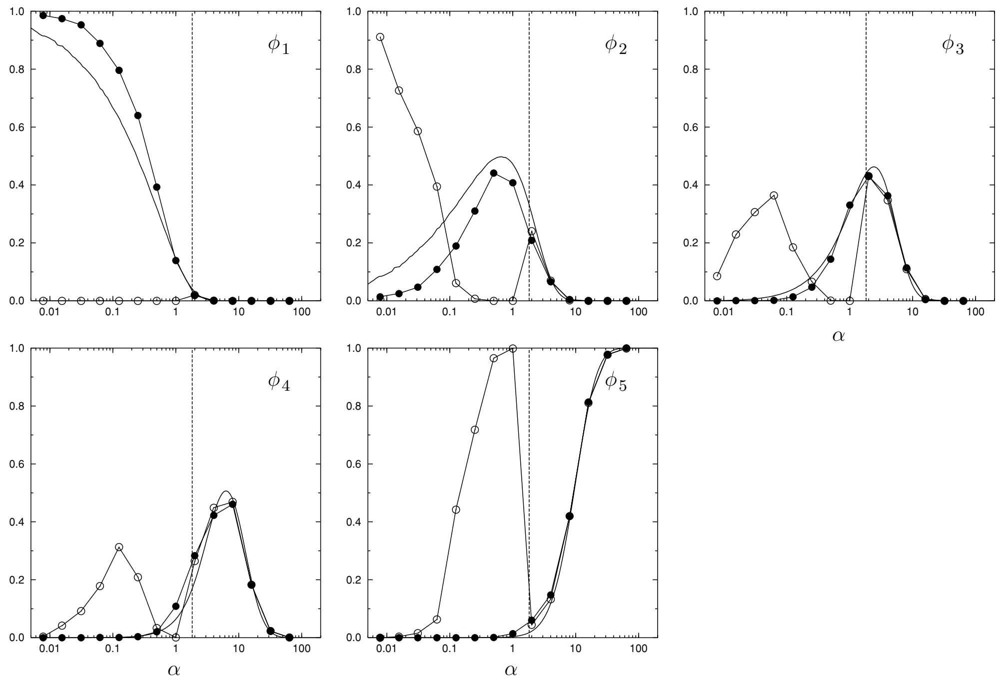

Figure 14. The fractions ϕ_ℓ for $S = 5$ are shown together with corresponding data measured in numerical simulations of MGs without decision noise, for biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, $\bullet$) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, $\circ$). Dashed: the phase transition point $\alpha_c(5) \approx 1.822$.

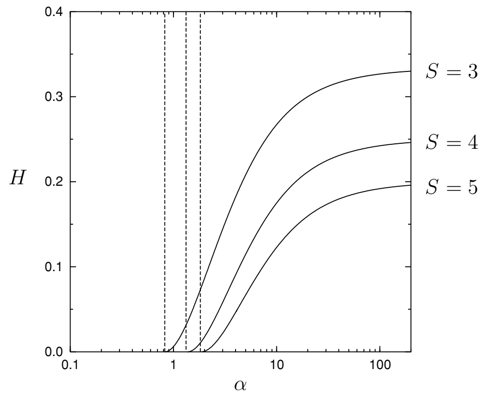

Figure 15. The measure $H = \lim_{N\to\infty} p^{-1}\sum_\mu[\lim_{\tau\to\infty}\tau^{-1}\sum_{t=1}^{\tau}(A_\mu(t) - \langle A(t)\rangle)]^2$ of the overall market bid predictability, for time-translation invariant states, as a function of the control parameter $\alpha = p/N$ for $S \in \{3, 4, 5\}$ (from top to bottom). Dashed: the corresponding phase transition points $\alpha_c(S)$.

This is still exact, but now we approximate for $t \neq t'$

$$\sum_a \langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}\rangle_{\ell\,\text{active}} \approx \sum_a \langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\rangle_{\ell\,\text{active}} \times \langle \delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}\rangle_{\ell\,\text{active}} \approx \ell^{-1}.$$

This gives $\tilde{C}(t-t') \approx \delta_{tt'} - c + (1-\delta_{tt'})\sum_{\ell=1}^{S}\phi_\ell/\ell$ and hence the approximation

$$\sigma_B^2 = \frac{\sum_{\ell=1}^{S}\phi_\ell/\ell}{(1+\tilde{\eta}\chi)^2} + 1 - \sum_{\ell=1}^{S}\phi_\ell/\ell. \tag{76}$$

Formula (74) is identical to what was already proposed for $S = 2$ in [6]; formula (76) generalizes to arbitrary values of S the $S = 2$ volatility approximation first published in [10, 11]. Our two approximation formulae have more or less opposite deficits, since (74) fails to incorporate any transient contributions to the correlations (here $\tilde{C}(t) = 0$ for $t \neq 0$), whereas (76) incorporates too many (here $\lim_{t\to\infty}\tilde{C}(t) = \sum_{\ell=1}^{S}\phi_\ell/\ell - c \neq 0$). For batch MGs one finds that (76) is more accurate than (74), see e.g., figure 16.

11. Dynamics for very small and very large α

Solving our order parameters C and G from (21) for finite times is non-trivial due to the non-Markovian nature of the effective process (19). Only for extreme values of α, viz. for $\alpha \to 0$ and $\alpha \to \infty$, and for short times it can to some extent be done. The causality constraint $G_{tt'} = 0$ for $t \geqslant t'$ allows us to calculate with little difficulty the first few time steps and gain a better understanding of the nature of the dynamics, especially with regard to the role of the initial conditions and the dependence on the control parameter α. Causality implies that $(G^n)_{tt'} = 0$ for $t - t' < n$, so that for $t \geqslant t' \geqslant 0$:

$$R_{tt'} = \tilde{\eta}(\mathbb{1} + \tilde{\eta}G)^{-1}_{tt'} = \tilde{\eta}\sum_{n=0}^{t-t'}(-\tilde{\eta})^n(G^n)_{tt'} \tag{77}$$

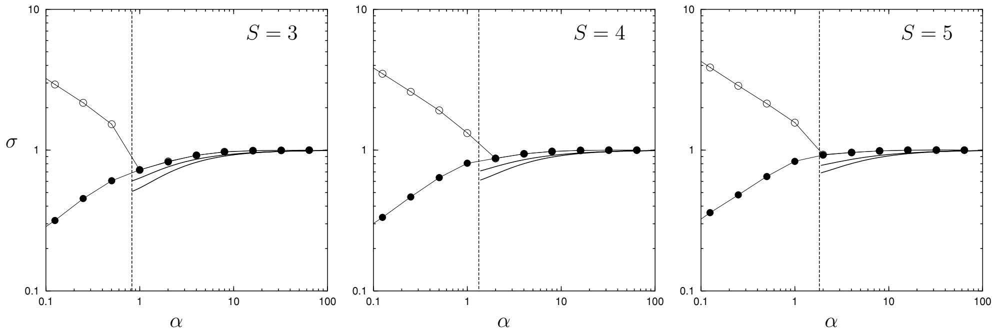

Figure 16. The two approximate volatility formulae σ_A and σ_B (lower and upper solid curves, respectively) as functions of $\alpha = p/N$, for $S \in \{3, 4, 5\}$, calculated in the ergodic regime $\alpha > \alpha_c$. They are shown together with volatility data measured in numerical simulations of MGs without decision noise, for both biased initial conditions (random initial strategy valuations drawn from $[-10, 10]$, •) and for unbiased initial conditions (random initial strategy valuations drawn from $[-10^{-4}, 10^{-4}]$, ∘). Simulation system size: $N = 4097$. Dashed: the corresponding phase transition points $\alpha_c(S)$.

$$\begin{aligned}\Sigma_{tt'} &= \langle \eta_a(t)\eta_b(t')\rangle = \delta_{ab}(RCR^{\dagger})_{tt'} \\ &= \frac{\delta_{ab}}{\tilde{\eta}^2}\sum_{s=0}^{t}\sum_{s'=0}^{t'}\sum_{n=0}^{t-s}\sum_{n'=0}^{t'-s'}(-\tilde{\eta})^{n+n'}(G^n)_{ts}(G^{n'})_{t's'}C_{ss'}.\end{aligned} \tag{78}$$

For short times all these expressions reduce to simple finite sums; for instance:

$$\begin{aligned} &R_{00} = R_{11} = \tilde{\eta}, && \Sigma_{00} = \tilde{\eta}^2 \\ &R_{10} = -\tilde{\eta}^2 G_{10}, && \Sigma_{10} = \tilde{\eta}^2(C_{10} - G_{10}) \\ &R_{21} = -\tilde{\eta}^2 G_{21}, && \Sigma_{11} = \tilde{\eta}^2 - 2\tilde{\eta}^3 G_{10}C_{10} + \tilde{\eta}^4 (G_{10})^2 \\ &R_{20} = \tilde{\eta}^3 G_{21}G_{10} - \tilde{\eta}^2 G_{20}. \end{aligned}$$

One can now in the familiar manner solve the effective single agent process for the first few time steps. The resulting expressions would be fully exact (for any α, including those in the nonergodic regime $\alpha < \alpha_c(S)$), but increasingly involved.

For very large and very small α matters become relatively simple, but the result is still quite informative. We will discuss only the deterministic case; adding decision noise will only introduce transparent stochastic variations of the behaviour described below. For large α and finite times the single agent process tells us that $G = \mathcal{O}(1/\tilde{\eta}\sqrt{\alpha})$, that the effective Gaussian noise scales as $\eta_a(t) = \mathcal{O}(1)$, and that

$$t = 0: \quad v_a(1) = -\alpha\delta_{a,\mathrm{argmax}_b[v_b(0)]} + \mathcal{O}(\sqrt{\alpha}) \tag{79}$$

$$t > 0: \quad v_a(t+1) = v_a(t) - \alpha\delta_{a,\mathrm{argmax}_b[v_b(t)]} + \mathcal{O}(\sqrt{\alpha}). \tag{80}$$

Let us assume that $v_a(0) = \mathcal{O}(\alpha^0)$ for all α, and exclude the pathological case where two or more initial valuations are identical. According to (79), the strategy with the largest initial valuation is selected at time $t = 1$, but will then move to the back of the list of ordered valuations. At the next step the second largest is selected, and then, following (80), also moved to the back of the ordered list, etc. The net result is that all valuations $v_a(t)$ will become increasingly negative (proportional to α), growing on average linearly with time, and that the effective agent will continually alternate his S strategies in a *fixed* order, being the order in which the valuations are ranked initially. If, for example, $v_1(0) > v_2(0) > \cdots > v_S(0)$, then $\mathrm{argmax}_b[v_b(t)] = t + 1 \bmod S$, and for $t > 0$ the effective agent equation gives

$$v_a(t) = -\alpha \sum_{t'=0}^{t-1} \delta_{a,t'+1 \bmod S} + \mathcal{O}(t\sqrt{\alpha})$$
$$= -\alpha \,\mathrm{int}\left[\frac{S+t+1-a}{S}\right] + \mathcal{O}(t\sqrt{\alpha}), \tag{81}$$

where $\mathrm{int}[z]$ denotes the largest integer m such that $z \geqslant m$. This is the generalization to $S > 2$ of the period-2 oscillations known to occur in MGs with $S = 2$. For arbitrary valuation initializations $\mathbf{v}(0) = \mathbf{v}_0$ the above solution generalizes to

$$v_a(t) = -\alpha \,\mathrm{int}\left[\frac{S+t+1-\pi_{\mathbf{v}_0}(a)}{S}\right] + \mathcal{O}(t\sqrt{\alpha}) \tag{82}$$

with $\pi_{\mathbf{v}}$ denoting that permutation of $\{1, \ldots, S\}$ for which $v_{\pi(1)} > v_{\pi(2)} > \cdots > v_{\pi(S)}$. If $\mathbf{v}_0$ is drawn randomly from some finite-width distribution, corresponding to the situation where the agents in the original N-agent system are initialized non-identically, all agents would still continually alternate their strategies in a fixed order, but the orders would now generally be different for different agents. The dynamic order parameters would in either case be $C_{tt'} = \delta_{t,t' \bmod S} + \mathcal{O}(\alpha^{-1/2})$ and $G_{tt'} = \mathcal{O}(\alpha^{-1/2})$.

Let us finally turn to small α. Here the process (19) describes only small valuation changes at each time step, and one consequently finds that $C_{tt'} = 1 + \mathcal{O}(\sqrt{\alpha})$ and $G_{tt'} = \mathcal{O}(\sqrt{\alpha})$. The Gaussian noise in (19) is static in leading order in α, $\langle \eta_a(t)\eta_b(t')\rangle = \tilde{\eta}^2\delta_{ab} + \mathcal{O}(\sqrt{\alpha})$, and in the absence of perturbation fields equation (19) gives simply

$$v_a(t) = v_a(0) + t\tilde{\eta}\sqrt{\alpha}z_a + \mathcal{O}(\alpha), \tag{83}$$

where the z_a are independent frozen random Gaussian variables, with zero average and unit variance. The systems remain static for a period of order $t \sim \Delta/\tilde{\eta}\sqrt{\alpha}$, where Δ indicates the magnitude of the initial valuation differences. A full analysis of the solution of our effective agent equations following this transient stage is in the small α regime a highly nontrivial exercise, which (to our knowledge) even for $S = 2$ has not yet been carried out, and would merit a full and extensive study in itself.

12. Discussion

In this paper we have shown how the generating functional analysis theory of minority games, developed in full initially only for $S = 2$, can be generalized to MGs with arbitrary values of S. The key obstacle in this generalization turned out not to be the derivation of closed equations for dynamic order parameters (via a generalized effective single agent process) but rather the solution of these equations in time-translation invariant stationary states. In previous studies closure of persistent order parameter equations could not yet be achieved analytically, and equations had to be closed artificially with the help of simulation data [6, 8]. At a technical level the basic problem was the calculation of the strategy selection frequencies of the effective agent. This problem has now been solved, resulting in exact and explicit closed equations for persistent order parameters and for phase transition points, for any value of S.

In our applications of the resulting theory we have mainly concentrated on the simplest nontrivial case $S = 3$, complemented by further applications to $S = 4$ and $S = 5$. In all cases, the predictions of our theory in time translation invariant stationary states without anomalous response were shown to agree perfectly with numerical simulation data, including sensitive measures such as the strategy frequency distribution. We have not been able to solve our order parameter equations in all possible situations, however, those regimes where we could not proceed to full solution (e.g., calculating stationary states in the regime $\alpha < \alpha_c(S)$, and non-persistent order parameters at arbitrary times) are the same as those which also for the simpler case $S = 2$ have so far resisted the efforts of statistical mechanics. Put differently, our objective and contribution here has been to raise the solvability of MGs with arbitrary values of S to the same level as that of MGs with $S = 2$.

It will clear that several further applications, developments and generalizations of the theory could now be taken up. One could for instance explore in more detail the effects of decision noise on MGs with $S > 2$, for which we have generated the required mathematical tools but for which we have not worked out the full consequences (such as the often counter-intuitive impact on the volatility, or the phase diagrams for multiplicative noise in the (α, T) plane). Alternatively, one could develop an $S > 2$ generating functional analysis for the so-called fake history on-line MGs [13], where valuation updates are made after each randomly drawn sample of the global information. Probably the most interesting and nontrivial next step, however, would be to investigate the structure and the stationary state solutions of an $S > 2$ theory for MGs where the global information is no longer drawn randomly but represents the actual global history of the market, by the generalization of [14].

Appendix A. Derivation of effective single agent equation

Extremization of the exponent $\Psi + \Phi + \Omega$, as defined by (16)–(18), with respect to the dynamic order parameters $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$ gives the following saddle-point equations:

$$C_{tt'} = \sum_a \langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))} \delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))} \rangle_\star \tag{A.1}$$

$$K_{tt'} = \sum_a \langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))} \hat{v}_a(t') \rangle_\star \tag{A.2}$$

$$L_{tt'} = \sum_a \langle \hat{v}_a(t) \hat{v}_a(t') \rangle_\star \tag{A.3}$$

$$\hat{C}_{tt'} = \mathrm{i}\frac{\partial \Phi}{\partial C_{tt'}} \qquad \hat{K}_{tt'} = \mathrm{i}\frac{\partial \Phi}{\partial K_{tt'}} \qquad \hat{L}_{tt'} = \mathrm{i}\frac{\partial \Phi}{\partial L_{tt'}} \tag{A.4}$$

with the abbreviation $\langle f(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) \rangle_\star = \lim_{N\to\infty} N^{-1} \sum_i \langle f(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) \rangle_i$, where

$$\langle f(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) \rangle_i = \frac{\int \left[\prod_{at} \frac{\mathrm{d}v_a(t)\,\mathrm{d}\hat{v}_a(t)}{2\pi}\right] \langle f(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) F_i(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) \rangle_{\mathbf{z}}}{\int \left[\prod_{at} \frac{\mathrm{d}v_a(t)\,\mathrm{d}\hat{v}_a(t)}{2\pi}\right] \langle F_i(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) \rangle_{\mathbf{z}}} \tag{A.5}$$

$$\begin{aligned} F_i(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) = P_0(\mathbf{v}(0)) \exp\left(\mathrm{i} \sum_{at} \hat{v}_a(t)[v_a(t+1) - v_a(t) - \theta_{ia}(t)] + \mathrm{i} \sum_{at} \psi_{ia}(t) \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))} \right) \\ \times \exp\Bigg(-\mathrm{i} \sum_{att'} \big[\hat{C}_{tt'} \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))} \delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))} + \hat{L}_{tt'} \hat{v}_a(t) \hat{v}_a(t') \\ + \hat{K}_{tt'} \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))} \hat{v}_a(t') \big] \Bigg). \end{aligned} \tag{A.6}$$

Via (9), (10) and $\overline{Z[\mathbf{0}]} = 1$ one confirms as usual that the $C_{tt'}$ are the correlations in (9), that $L_{tt'} = 0$, and that $K_{tt'} = iG_{tt'}$. Putting $\psi \to \mathbf{0}$ and choosing $\theta_{ia} = \theta_a$ (site-independent perturbations) eliminates the dependence of (A.6) on i: $F_i(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z}) = F(\mathbf{v}, \widehat{\mathbf{v}}, \mathbf{z})$. Next, to evaluate (A.4) we work out the function Φ (17) for small $\{L_{tt'}\}$. Upon eliminating K via $K = iG$, and with the short-hands $\mathbb{1}$ for the unit matrix and $(A^\dagger)_{tt'} = A_{t't}$, we find

$$\Phi = -\frac{1}{2}\alpha \log\det[(\mathbb{1}+\tilde{\eta}G^\dagger)(\mathbb{1}+\tilde{\eta}G)] - \frac{1}{2}\alpha\tilde{\eta}^2 \sum_{tt'} L_{tt'}[(\mathbb{1}+\tilde{\eta}G)^{-1}C(\mathbb{1}+\tilde{\eta}G^\dagger)^{-1}]_{tt'} + \mathcal{O}(L^2). \tag{A.7}$$

For $L = 0$ the three saddle-point equations (A.4) now become

$$\hat{C}_{tt'} = 0 \qquad \hat{K}_{tt'} = -\alpha\tilde{\eta}(1+\tilde{\eta}G^\dagger)^{-1}_{tt'} \tag{A.8}$$

$$\hat{L}_{tt'} = -\tfrac{1}{2}\mathrm{i}\alpha\tilde{\eta}^2[(\mathbb{1}+\tilde{\eta}G)^{-1}C(\mathbb{1}+\tilde{\eta}G^\dagger)^{-1}]_{tt'}. \tag{A.9}$$

Upon inserting these expressions into (A.6), and using causality, one can now prove that the denominator of (A.5) equals one. This, in turn, implies that $\langle g(\mathbf{v},\mathbf{z})\hat{v}_a(t)\rangle_\star = \mathrm{i}\partial\langle g(\mathbf{v},\mathbf{z})\rangle_\star/\partial\theta_a$. We can then integrate out all occurrences of the conjugate integration variables $\{\hat{v}_a\}$, and end up with the remaining saddle-point equations:

$$C_{tt'} = \sum_a \langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}\rangle_\star \tag{A.10}$$

$$G_{tt'} = \sum_a \frac{\partial}{\partial\theta_a(t')}\langle \delta_{a,m(\mathbf{v}(t),\mathbf{z}(t))}\rangle_\star, \tag{A.11}$$

where

$$\langle f(\mathbf{v},\mathbf{z})\rangle_\star = \int \left[\prod_{at} \mathrm{d}v_a(t)\right] \langle f(\mathbf{v},\mathbf{z})F(\mathbf{v},\mathbf{z})\rangle_{\mathbf{z}} \tag{A.12}$$

$$\begin{aligned} F(\mathbf{v},\mathbf{z}) = P(\mathbf{v}(0)) \int \left[\prod_{at}\frac{\mathrm{d}\eta_a(t)}{\sqrt{2\pi}}\right] &\times \prod_a \left\{ \frac{\exp\left(-\frac{1}{2}\tilde{\eta}^{-2}\sum_{tt'}\eta_a(t)[(\mathbb{1}+\tilde{\eta}G^\dagger)C^{-1}(\mathbb{1}+\tilde{\eta}G)]_{tt'}\eta_a(t')\right)}{\det^{-\frac{1}{2}}[(\mathbb{1}+\tilde{\eta}G^\dagger)C^{-1}(\mathbb{1}+\tilde{\eta}G)]} \right\} \\ &\times \prod_{at}\left[v_a(t+1) - v_a(t) - \theta_a(t) + \alpha\tilde{\eta}\sum_{t'}(\mathbb{1}+\tilde{\eta}G)^{-1}_{tt'}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))}\right]. \end{aligned} \tag{A.13}$$

We recognize the above measure to represent the statistics of an effective single agent process, with dynamics defined as

$$v_a(t+1) = v_a(t) + \theta_a(t) - \alpha\tilde{\eta}\sum_{t'} R_{tt'}\delta_{a,m(\mathbf{v}(t'),\mathbf{z}(t'))} + \sqrt{\alpha}\eta_a(t). \tag{A.14}$$

Here $R_{tt'} = \tilde{\eta}(\mathbb{1}+\tilde{\eta}G)^{-1}_{tt'}$, and $\eta_a(t)$ is a Gaussian noise characterized by the moments $\langle\eta_a(t)\rangle = 0$ and $\langle\eta_a(t)\eta_b(t')\rangle = \delta_{ab}(RCR^\dagger)_{tt'}$.

Appendix B. The volatility matrix

Here we outline briefly how the generating functional (22) for overall bid fluctuations can be calculated via simple modifications of the generating functional $\overline{Z[\psi]}$ defined in (8). Comparison with (8) shows that in the latter we should replace

$$\exp\left(\mathrm{i}\sum_{iat}\psi_{ia}(t)\delta_{a,m(\mathbf{v}_i(t),\mathbf{z}_i(t))}\right) \to \exp\left(\mathrm{i}\sum_{\mu at}\phi_\mu(t)N^{-1/2}\sum_i R^{ia}_\mu\delta_{a,m(\mathbf{v}_i(t),\mathbf{z}_i(t))}\right).$$

This implies making the replacement $x_t^\mu \to x_t^\mu + \phi_\mu(t)$ in the disorder average, and affects only on the exponent Φ of the saddle-point problem. The latter becomes

$$\Phi = -\frac{1}{2}\alpha \log\det[(\mathbb{1} + \tilde{\eta}G^\dagger)(\mathbb{1} + \tilde{\eta}G)] \\ + \frac{1}{N}\sum_\mu \log\left[1 - \frac{1}{2}\sum_{tt'}\left(L_{tt'} + \frac{\phi_\mu(t)\phi_\mu(t')}{\tilde{\eta}^2}\right)(RCR^\dagger)_{tt'} + \mathcal{O}(L^2, \phi^4)\right]. \tag{B.1}$$

We can now calculate from $\overline{Z[\phi]}$ the quantities of interest. Upon emphasizing the dependence of (B.1) on the fields ϕ, and using the normalization $\overline{Z[\mathbf{0}]} = 1$, we obtain

$$\begin{aligned}\lim_{N\to\infty}\overline{\langle A_\mu(t)\rangle} &= -\mathrm{i}\lim_{N\to\infty}\lim_{\psi\to\mathbf{0}}\frac{\partial\overline{Z[\phi]}}{\partial\phi_\mu(t)} \\ &= -\mathrm{i}\lim_{\psi\to\mathbf{0}}\frac{\partial}{\partial\phi_\mu(t)}\prod_\lambda\left[1 - \frac{1}{2}\sum_{ss'}\phi_\lambda(s)\frac{(RCR^\dagger)_{ss'}}{\tilde{\eta}^2}\phi_\lambda(s') + \cdots\right] = 0\end{aligned} \tag{B.2}$$

$$\begin{aligned}\lim_{N\to\infty}\overline{\langle A_\mu(t)A_\nu(t')\rangle} &= -\lim_{N\to\infty}\lim_{\phi\to\mathbf{0}}\frac{\partial^2\overline{Z[\phi]}}{\partial\phi_\mu(t)\partial\phi_\nu(t')} \\ &= -\lim_{\psi\to\mathbf{0}}\frac{\partial^2}{\partial\phi_\mu(t)\phi_\nu(t')}\prod_\lambda\left[1 - \frac{1}{2}\sum_{ss'}\phi_\lambda(s)\frac{(RCR^\dagger)_{ss'}}{\tilde{\eta}^2}\phi_\lambda(s') + \cdots\right] \\ &= \tilde{\eta}^{-2}\delta_{\mu\nu}(RCR^\dagger)_{tt'}.\end{aligned} \tag{B.3}$$

Appendix C. Integration identities

Here we simply list (without proof) some of the basic identities that one uses in doing the various integrals in the $S = 3$ theory analytically, for the benefit of the reader:

$$\int_0^u \mathrm{D}x\, x^2 = \frac{1}{2}\operatorname{erf}\left(\frac{u}{\sqrt{2}}\right) - \frac{u}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}u^2} \tag{C.1}$$

$$\int \mathrm{D}x\operatorname{erf}(A + Bx) = \operatorname{erf}\left(\frac{A}{\sqrt{1+2B^2}}\right) \tag{C.2}$$

$$\int \mathrm{D}x\, x\operatorname{erf}(A + Bx) = \frac{2B}{\sqrt{\pi(1+2B^2)}}\,\mathrm{e}^{-A^2/(1+2B^2)} \tag{C.3}$$

$$\int \mathrm{D}x\operatorname{erf}^2(Bx) = \frac{4}{\pi}\arctan\left(\sqrt{1+4B^2}\right) - 1 \tag{C.4}$$

$$\int \mathrm{D}x\operatorname{erf}^2(A + Bx) = \frac{4}{\pi}\arctan\left(\sqrt{1+4B^2}\right) - 1 + \frac{4}{\sqrt{\pi}}\int_0^{A/\sqrt{1+2B^2}}\mathrm{d}x\,\mathrm{e}^{-x^2}\operatorname{erf}\left(\frac{x}{\sqrt{1+4B^2}}\right). \tag{C.5}$$

Note added. While finishing this paper we were made aware of another study in progress, aiming also to solve the strategy frequency problem for minority games with more than two strategies per agent, but in the context of multi-asset MGs [15], and using a somewhat different approach (which one must ultimately expect to be mathematically equivalent).

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